Time Consistency and Intergenerational Risk Sharing

By Tim Worrall

Keele University

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Abstract

It is shown how intergenerational risk sharing can be achieved by transfers from the young generation to the old generation such that the young generation will never have an incentive to unilaterally renege on the transfer. This contradicts a claim made in Gordon and Varian (1988) that intergenerational risk-sharing is infeasible because of problems of time consistency. It is shown however, that even in a stationary environment, time consistent transfers are always non-stationary.

Keywords: Intergenerational risk-sharing; Social compact; Time consistency; Self-enforcing.

JEL Codes: D91; H55.

1. Introduction

Economic efficiency dictates that where possible risk be shared. Any single generation can share idiosyncratic but not aggregate risk. Aggregate or social risk for one generation is however, idiosyncratic when compared with the social risks of different generations. Therefore there is a potential benefit to sharing risk between generations.

A large literature has explored these potential benefits from intergenerational risk sharing using a stochastic overlapping generations framework. This literature has considered ways in which both *interim* and *ex ante* efficiency can be enhanced through market mechanisms and pay-as-you-go social security schemes under a variety of different assumptions about production possibilities and the stochastic processes generating risk.[1] In addition a number of papers, such as Krueger and

[1] See e.g. Weiss (1980), Enders and Lapan (1982), Smith (1982), Gordon and Varian (1988), Hassler and Lindbeck (1997), Thøgersen (1998), Marini and Scaramozzino (1999), Demanage (2002), Thøgersen (2003) and Demanage (2005).

Kubler (2006) and Gottardi and Kubler (2006), have used models of this type to estimate the welfare gains from social security reform by improving risk sharing across generations.

Despite the success of this literature less attention has been paid to the issue of the time consistency of any social security scheme. Whilst it might be possible to devise risk sharing programs such that every generation benefits from the scheme at the time of birth, *ex post* there may be states where agents will not voluntarily contribute to the system. If such states occur then the scheme is not time consistent. One reason why less attention has been paid to the time consistency problem is that in their seminal article Gordon and Varian (1988) came to the stark conclusion that "*an intergenerational risk-sharing scheme is infeasible due to problems of time consistency*".[2] Thus they conclude that without some commitment intergenerational risk sharing is impossible.

The impossibility of time consistent intergenerational risk sharing however, sits somewhat uneasily with the strategic "pension games" considered by Hammond (1975) and the folk theorem conclusions of non-stochastic overlapping generation games, e.g. Crémer (1986), Kandori (1992) and Smith (1992). Consider, for example, the case where the old and young play a prisoners' dilemma game. The old will always defect as there is no future payoff. Nevertheless the young may play co-operate in the expectation that the future young will also play co-operate against them when they are old. If the young deviate to defect, then subsequent play is for all future players to defect at each period, which is the one-shot Nash equilibrium. If the discount factor is large enough, the threat is credible and the players will prefer to co-operate when young. In the limit if the discount factor is close enough to one, then a folk theorem result emerges that any individually rational payoffs are sustainable as a sub-game perfect equilibrium. With this interpretation social security might arise from a social compact between the generations and is enforced by a threat of breakdown in cooperation without a need for any external commitment devise.

This paper shows that these two strands of the literature are not inconsistent. Intergenerational risk sharing is possible through a purely social compact between generations. Section 4 shows that Gordon and Varian (1988) were correct in as-

[2] Gordon and Varian (1988), p.195.

serting that there can be no time consistent intergenerational risk sharing scheme provided that transfers are *stationary*. However, the pension games approach suggests that by relaxing the assumption of stationarity and allowing the risk sharing scheme to exhibit *history dependence* it will be possible to share risk across generations and achieve time consistency. Section 5 confirms this supposition. Since the purpose here is only to show existence of time consistent intergenerational risk sharing, Section 5 constructs an example using a simple variant of the Gordon-Varian model. Moreover, it shows that it is possible to construct time consistent intergenerational risk sharing schemes that are actuarially fair. In these actuarially fair schemes the expected transfer of the young have a martingale property so that on average each young generation pays into the scheme the same amount as the immediately proceeding generation.

The paper proceeds as follows. Section 2 presents the Gordon-Varian model. Section 3 considers feasible transfer schemes when commitment exists. Section 4 considers the time consistency issue and Section 5 constructs examples of time consistent risk sharing schemes. Section 6 discusses possible directions for future research.

2. The Simple Gordon-Varian Model

Gordon and Varian (1988) provided a simple and powerful overlapping generations model of intergenerational risk sharing which has been used extensively in subsequent work such as Thøgersen (1998) and Marini and Scaramozzino (1999). Here we present the simplest version of the Gordon-Varian model. Agents live for two periods. They earn a non-random wage, w when young and consume only when old. If the young agent has an income of w, then it is saved and produces a return of $w - \varepsilon$,[3] where ε is a random variable with mean, $\mathrm{E}[\varepsilon] = 0$ and variance, $\mathrm{Var}[\varepsilon] = \mathrm{E}[\varepsilon^2] = \sigma^2 > 0$. The realizations of ε is the same for all members of the generation and occurs after the previous generation has died but before the new generation is born, so that all risk in the economy at any one point in time is aggregate risk. There is no population growth. It is assumed that utility when old depends on the mean and variance of consumption according to a mean-variance

[3] This is the simplest model presented in Gordon and Varian (1988) but with the sign on ε reversed.

utility function $U = \mathrm{E}[c] - \mathrm{Var}[c]$,[4] where $c = w - \varepsilon$ is the random old age consumption. The ε are *i.i.d.* random variables, so that absent any transfers, each generation has an autarky utility of $w - \sigma^2$.

3. With Commitment

To examine intergenerational risk sharing suppose that it is possible to make a transfer x_t from the young alive at time t to the old alive at time t. The following Ponzi-type scheme of transfers works quite well. Let $x_1 = \varepsilon_1$. The consumption of the old at time $t = 1$ is therefore w with a utility of w since the randomness is completely eradicated. The return of the old at time $t = 2$ is $w - x_1 - \varepsilon_2 = w - \varepsilon_1 - \varepsilon_2$, therefore let $x_2 = \varepsilon_1 + \varepsilon_2$ so that again consumption is w with utility of w. In general

$$x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \varepsilon_t = \sum_{i=1}^{t} \varepsilon_i & \text{For } t \geq 1. \end{cases}$$

Consumption of the old alive at time t is $c_t = w - x_{t-1} - \varepsilon_t + x_t = w$. Essentially since risk is all idiosyncratic across the generations it can be effectively eliminated by pooling over all generations, *all good or bad luck is transferred to the future*. One problem with this scheme is that it does not respect the non-negativity constraint on the income of the young at date t:

$$y_t = w - x_t \geq 0 \qquad \forall t. \tag{1}$$

To see this note that although $\mathrm{E}[x_t] = 0$, $\mathrm{Var}[x_t] = t\sigma^2$, so that the variance is unbounded and no matter how large is w, (1) will be violated with probability one in finite time.

The scheme actually considered by Gordon-Varian is a variant of the above scheme that has equal risk-sharing between $n \geq 2$ generations.

$$x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{t-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t < n \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{n-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t \geq n. \end{cases}$$

[4] This is the standard constant absolute risk aversion mean-variance utility function $U = \mathrm{E}[c] - \frac{1}{2} A \,\mathrm{Var}[c]$, with $A = 2$

Then consumption of the old alive at time t is $c_t = w - x_{t-1} - \varepsilon_t + x_t = w - \sum_{i=0}^{n-1} \frac{\varepsilon_{t-i}}{n}$ with $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \frac{\sigma^2}{n}$ for $t \geq n$. The utility of the old alive at time t is $u = w - \frac{\sigma^2}{n}$. The scheme works best when n is large and as $n \to \infty$ utility tends to the first-best level. Again however there is a conflict with the inequality (1). The variance of the transfer is given by[5]

$$\mathrm{Var}[x_t] = \frac{n-1}{n}\frac{2n-1}{6}\sigma^2.$$

This is increasing without bound in n so that as n is increased (1) is violated, and hence the scheme will become infeasible, with probability one in finite time.

Another simple way to bound the variance of the transfer (apart from choosing n small) is to have less risk-sharing over time. A simple scheme that does this is

$$x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \frac{\varepsilon_t}{\sqrt{2^{t-1}}} & \text{For } t \geq 1. \end{cases}$$

This offers complete insurance at date $t = 1$ as $x_1 = \varepsilon_1$. Consumption at time t is $c_t = w + \varepsilon_t(1 + \frac{1}{\sqrt{2^{t-1}}})$, so $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \sigma^2(1 - 2(2^{\frac{1-t}{2}} - 2^{-t}))$ which is increasing over time. The utility of the generation born at $t-1$ is $w - \sigma^2(1 - 2(2^{\frac{1-t}{2}} - 2^{-t}))$, which although declining over time toward the autarky level, provides a Pareto-improvement over autarky for each generation. Moreover the expected transfer is $\mathrm{E}[x_t] = 0$ and although the variance of the transfer is increasing over time, $\mathrm{Var}[x_t] \to 2\sigma^2$ as $t \to \infty$. Thus provided the support of ε is small relative to w the Pareto-improvement can be achieved without violating (1).

4. Time Consistency

In this section we consider the sequence of transfers as a kind of social compact in which any generation may freely renege if it is to their advantage. Let $h^t = (\varepsilon_1, \varepsilon_2, ...\varepsilon_t)$ be a history of shocks. A social compact is a sequence of transfers $(x_t(h^t))$ such that no generation will find it to their advantage to renege on any transfer $x_t(h^t)$. The constraint for the old generation at time t is

$$u(w - \varepsilon_t - x_{t-1}(h^{t-1}) + x_t(h^{t-1}, \varepsilon_t)) \geq u(w - x_{t-1}(h^{t-1})) \tag{2}$$

[5]This variance is calculated from the recurrence relation and the proof is given in the Appendix.

for all ε_t. It is clear that the old will never make a transfer to the young as they have nothing to gain from such a transfer, so we must have

$$x_t(h^t) \geq 0 \qquad \forall h^t. \tag{3}$$

This is exactly as in the prisoners' dilemma game described in the introduction. The old never have an incentive to co-operate. Nevertheless the young may still have an incentive to make a transfer in anticipation of transfers to them by the future young. This is an important feature of the time consistent social compact, any compact has a contingent pension provision. Thus in the terms of Rangel and Zeckhauser (2001) all trade involves backward transfers with no forward transfers from old to young generations.

Notice that neither the Ponzi scheme nor the Gordon-Varian n generation scheme nor the the declining risk-sharing scheme discussed in Section 3 respect this constraint since they all involve forward transfers from the old to the young after some histories. None of these schemes is therefore time consistent.

Now suppose that any failure by the young at time t to make the appropriate transfer results in a breakdown of trust and the social compact dissolves thereafter, $x_\tau = 0 \quad \forall \tau > t$. Then clearly if the young at time t are to renege they will choose $x_t = 0$ and this is a Nash equilibrium exactly as in the prisoners' dilemma game. The young may have an incentive to make the current transfer if they anticipate that the young next period will also make a transfer next period if they suffer a bad shock and $\varepsilon > 0$. Consumption of the old at time $t+1$ is $c_{t+1}(h^{t+1}) = w - x_t(h^t) - \varepsilon_{t+1} + x_{t+1}(h^{t+1})$. For a given value of x_t, the constraint for the young at time t is

$$\mathrm{E}[u(w - \varepsilon_{t+1} - x_t(h^t) + x_{t+1}(h^t, \varepsilon_{t+1}))|h^t] \geq \mathrm{E}[u(w - \varepsilon_{t+1})]. \tag{4}$$

This equation says that the young at time t, knowing the shock ε_t and the called for transfer in that state, $x_t(h^{t-1}, \varepsilon_t)$, will prefer to make the transfer given what they can expect in their old age, rather than making no transfer and receiving no transfer in their old age in return.

The constraint of inequality (4) makes it clear that there can be no intergenerational risk sharing with stationary transfers. To see this suppose that the transfer

x depends only on the current shock, ε, say that $x = g(\varepsilon)$. Since the constraint must hold for any ε_t, it hold for $x_{max} = \max\{g(\varepsilon_t)\}$. But then for any monotone increasing function $u(w - \varepsilon_t - x_{max} + g(\varepsilon_t)) \leq u(w - \varepsilon_t)$ as $g(\varepsilon_t) - x_{max} \leq 0$. Thus if any intergenerational risk sharing can be achieved, it cannot be achieved through a stationary transfer scheme.

This makes calculation of the optimum non-stationary scheme difficult. So the purpose of this paper is only to show that it is possible to have time consistent intergenerational risk sharing. It is therefore sufficient to construct examples which show that such schemes exist. First we revert to the mean-variance framework of Gordon-Varian. Then the expected consumption at time $t+1$ is $\mathrm{E}[c_{t+1}] = w - x_t + \mathrm{E}[x_{t+1}]$ and $\mathrm{Var}[c_{t+1}] = \sigma^2 + \mathrm{Var}[x_{t+1}] - 2\,\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}]$. The young at time t will not renege on the transfer x_t provided

$$w - x_t + \mathrm{E}[x_{t+1}] - (\sigma^2 + \mathrm{Var}[x_{t+1}] - 2\,\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}]) \geq w - \sigma^2$$

or expressing this more simply, the young will not have an incentive to renege provided

$$x_t \leq \mathrm{E}[x_{t+1}] - (\mathrm{Var}[x_{t+1}] - 2\,\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}]) \qquad \forall t. \tag{5}$$

The inequalities (3) and (5) are the self-enforcing constraints for the old and young generations respectively. The mean variance formulation is convenient because if the social compact provides future insurance then $\mathrm{Cov}[x_{t+1}, \varepsilon_{t+1}] \geq 0$. Thus the inequality (5) demonstrates that the young will make the transfer when they expect the transfer next period to be high enough and when the future insurance offered is sufficiently great. A risk-sharing arrangement that is feasible, i.e. satisfies the non-negativity requirement (1), and is self-enforcing, i.e. satisfies both (3) and (5), is time consistent.

5. Example

In this section it is shown that it is possible to have a time consistent social compact of intergenerational risk-sharing. It is sufficient to consider a simple example. Suppose $\varepsilon \in \{-1, 1\}$ with equal probability and assume that $w > 1$. Three schemes will be considered; one which satisfies the constraints for the mean-

	$\varepsilon_t = -1$	$\varepsilon_t = 1$
$\varepsilon_{t-1} = -1$	$w+1$	w
$\varepsilon_{t-1} = 1$	w	$w-1$

Table 1: Consumption for two generation scheme

variance utility, one that satisfies the constraints for any strictly increasing utility function and one that satisfies the constraints for any risk averse utility function.

To start consider the two benchmarks of autarky and the Gordon-Varian two generation scheme. In autarky consumption is either $w-1$ or $w+1$ with equal probability. Hence expected consumption is $\mathrm{E}[c_t] = w$ and the variance of consumption is $\mathrm{Var}[c_t] = 1$. Expected utility is $w-1$. In the Gordon-Varian two generation scheme

$$x_t = x_{t-1} + \varepsilon_t - \frac{1}{2}(\varepsilon_t + \varepsilon_{t-1}); \qquad x_0 = 0;\ \ \varepsilon_0 = 0.$$

So $c_t = w - x_{t-1} - \varepsilon_t + x_t = w - \frac{1}{2}(\varepsilon_t + \varepsilon_{t-1})$. Thus consumption takes on one of four possible values shown in Table 1, each cell occurring with equal probability We have $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \frac{1}{2}$. So that utility is $w - \frac{1}{2}$, for each generation beyond the first (the first generation have utility $w - \frac{1}{4}$). Clearly this scheme does not satisfy the self-enforcing constraint, (3), e.g. $x_1 = \frac{1}{2}\varepsilon_1$ which is negative if $\varepsilon_1 = -1$. It does however satisfy the other two constraints. To check (5) we take x_{t-1} and ε_{t-1} as given, to find $\mathrm{E}[x_t] = x_{t-1} - \frac{1}{2}\varepsilon_{t-1}$, $\mathrm{Var}[x_t] = \frac{1}{4}$ and $\mathrm{Cov}[x_t, \varepsilon_t] = \frac{1}{2}$, so $\mathrm{E}[x_t] - (\mathrm{Var}[x_t] - 2\,\mathrm{Cov}[x_t, \varepsilon_t]) = x_{t-1} - \frac{1}{2}\varepsilon_{t-1} + \frac{3}{4}$. Thus (5) is satisfied as $\varepsilon_{t-1} \leq 1\frac{1}{2}$. With $n = 2$, $x_t = \pm\frac{1}{2}$, so that $\mathrm{E}[x_t] = 0$ and (1) is satisfied.

Since the mean-variance utility is non-monotonic it is actually possible to find a stationary scheme that is time consistent. To see this consider the following simple stationary scheme which does just as well as the two-generation scheme but satisfies the constraints (1), (3) and (5):

$$\begin{aligned} x_t &= 1 \qquad if \qquad \varepsilon_t = 1 \\ x_t &= 0 \qquad if \qquad \varepsilon_t = -1. \end{aligned} \tag{6}$$

This scheme is stationary because the transfer depends only on the current state and not the past history and satisfies (3) by construction. Constraint (1) is satisfied as x_t is at most 1 and by assumption $w > 1$. It also clearly satisfies the requirement (3). With $\mathrm{E}[x_t] = \frac{1}{2}$, $\mathrm{Var}[x_t] = \frac{1}{4}$ and $\mathrm{Cov}[x_t, \varepsilon_t] = \frac{1}{2}$, $\mathrm{E}[x_t] - (\mathrm{Var}[x_t] - 2\,\mathrm{Cov}[x_t, \varepsilon_t]) = \frac{1}{2} - (\frac{1}{4} - 1) = 1\frac{1}{4}$. Since x_{t-1} is at most 1, the constraint (5) is always satisfied. To be explicit, suppose that $\varepsilon_{t-1} = 1$ and the young at time $t-1$ are required to make a transfer of 1 to the old. If they do so, then their consumption next period will be either $w - 1$ if $\varepsilon_t = 1$ or w if $\varepsilon_t = -1$. Expected consumption is $w - \frac{1}{2}$ and the variance of consumption is $\frac{1}{4}$. Thus the utility at date t conditional on transferring 1 at date $t-1$ is $w - \frac{1}{2} - \frac{1}{4} = w - \frac{3}{4}$. This is to be compared with the autarky utility of not making the transfer and receiving no transfer at date t which yields a utility payoff of $w - 1$. Since x_t does not depend on x_{t-1} clearly the self-enforcing constraint (5) is also satisfied when no transfer is required at date $t-1$.

It is to be remembered that viewed ex ante x_{t-1} is also a random variable. So consumption $c_t = w - x_{t-1} - \varepsilon_t + x_t$ is the same as the two generation scheme given in Table 1. As each cell occurs with equal probability, $\mathrm{E}[c_t] = w$ and $\mathrm{Var}[c_t] = \frac{1}{2}$. Ex ante utility is $w - \frac{1}{2}$, for each generation. This is an improvement over autarky. The improvement however, hinges on the mean-variance framework. Conditional on $\varepsilon_{t-1} = 1$ consumption of the old under autarky in period t is $w - 1$ if $\varepsilon_t = 1$ and $w + 1$ if $\varepsilon_t = -1$. With the transfer scheme in (6), conditional on $\varepsilon_{t-1} = 1$ and the transfer being made in period $t-1$, consumption of the old in period t is $w - 1$ if $\varepsilon_t = 1$ and w if $\varepsilon_t = -1$. Thus autarky stochastically dominates the scheme (6) conditional on $x_{t-1} = 1$.

As has been shown in Section 4 a scheme like (6) will not work if the utility function is monotonically increasing. To show that the feasibility of an intergenerational risk sharing scheme does not depend on the mean-variance framework, we will now consider a scheme similar to the bounded scheme in Section 3 that satisfies the constraints (1), (3) and (5), i.e. is time consistent, and stochastically dominates autarky. First let $\varepsilon_t^+ = \max[0, \varepsilon_t]$ and let $\zeta_t = \sum_{i=1}^t \varepsilon_i^+$. Thus ζ_t counts the number of past bad shocks. Now consider the following scheme

$$\begin{array}{lcllll} x_t & = & x_{t-1} + \frac{1}{2^{\zeta_{t-1}+1}} & \quad if & \quad \varepsilon_t = 1 \\ x_t & = & x_{t-1} & \quad if & \quad \varepsilon_t = -1 \end{array} \tag{7}$$

with $x_0 = 0$. For $\zeta_t = z$, consumption is $c_t = w + 1$ if $\varepsilon = -1$ and $c_t = w - 1 + \frac{1}{2^z}$ if $\varepsilon = 1$. This stochastically dominates the autarky consumption in all periods although tends to the autarky allocation with probability one. Thus this scheme will satisfy the self-enforcing constraint for the young generation, (5) for any strictly increasing utility function. Note that the transfers in (7) are a supermartingale with $\mathrm{E}[x_t] \geq x_{t-1} \geq 0$ for every history h^t. The young make the transfer because more is promised in the future and bad luck is transferred to future generations.

The next scheme shows that is possible to have $\mathrm{E}[x_t] = x_{t-1}$ for almost all histories h^t. Such a scheme is actuarially fair as each generations expects to pay into the scheme exactly the same on average as the immediately proceeding generation. Starting from $x_0 = 0$, let

$$x_t = \max[0, x_{t-1} + \frac{\varepsilon_t}{2^t}] \tag{8}$$

In (8) if $x_t > 0$, then $\mathrm{E}[x_t] = x_{t-1}$ and $\mathrm{E}[c_t] = w$ and the variance of consumption is

$$\mathrm{Var}[c_t] = 1 - 2^{1-t}(1 - 2^{-1-t})$$

which is less than the autarky variance of 1. Thus for $x_t > 0$, the scheme has a mean preserving reduction in risk and for any risk averse utility function will be preferred to autarky. For histories where $x_t = 0$ the scheme works more like (7) with consumption undiminished at $w + 1$ when $\varepsilon = -1$ and raised above $w - 1$ when $\varepsilon = 1$. Thus although $\mathrm{E}[x_t] = x_{t-1}$ along almost all histories, there is still growth in x_t. This has to be the case for any risk sharing scheme which satisfies (3) starting from $x_0 = 0$.

6. Conclusion

It has been shown that it is possible to have some intergenerational risk sharing even when each generation can unilaterally renege on any risk sharing transfers if it is to their own advantage. This contradicts the claim made in Gordon and Varian (1988) that in these circumstances no intergenerational risk sharing is possible. Further it has been shown that any time consistent intergenerational risk sharing must take the form of a non-stationary state contingent pension payable by the young to the old generation if the utility function is monotonically increasing.

Future work should be directed at finding optimum time consistent schemes and studying more general stochastic overlapping generations models with growth and capital accumulation.

Appendix

Proof that $\text{Var}[x_t] = \frac{n-1}{n}\frac{2n-1}{6}\sigma^2$

The rule for determination of the transfer is:

$$x_t = \begin{cases} 0 & \text{For } t = 0 \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{t-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t < n \\ x_{t-1} + \varepsilon_t - \sum_{i=0}^{n-1} \frac{\varepsilon_{t-i}}{n} & \text{For } t \geq n. \end{cases}$$

Repeated substitutions to give

$$x_t = \sum_{i=0}^{n-1} \frac{n-1-i}{n} \epsilon_{t-i}$$

or writing it out in full

$$x_t = \frac{n-1}{n}\epsilon_t + \frac{n-2}{n}\epsilon_{t-1} + \frac{n-3}{n}\epsilon_{t-2} + \cdots$$
$$+ \frac{(n-1)-(n-2)}{n}\epsilon_{t-(n-2)} + \frac{(n-1)-(n-1)}{n}\epsilon_{t-(n-1)}.$$

Since the $\text{Var}[\epsilon_t] = \sigma^2$ and $Cov[\epsilon_{t-i}, \epsilon_{t-j}] = 0$ for $i \neq j$, the variance of x_t is determined by the sum of the squares of the coefficients. Thus it is necessary to calculate the partial sum

$$s_n = 1^2 + 2^2 + 3^2 + \cdots + (n-2)^2 + (n-1)^2.$$

One simple way to calculate this sum is to use the method of difference sequences which shows that

$$s_n = 1\binom{n}{2} + 2\binom{n}{3}.$$

Hence

$$s_n = \frac{n(n-1)}{2} + \frac{2n(n-1)(n-2)}{6} = \frac{n(n-1)(2n-1)}{6}.$$

Since the sum of the squares of the coefficients is $\frac{s_n}{n^2}$ we have

$$\text{Var}[x_t] = \frac{s_n}{n^2}\sigma^2 = \frac{(n-1)}{n}\frac{(2n-1)}{6}\sigma^2.$$

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