STOCHASTIC STABILITY OF MONOTONE ECONOMIES IN REGENERATIVE ENVIRONMENTS

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Abstract

We introduce and analyse a new class of monotone stochastic recursions in a regenerative environment which is essentially broader than that of Markov chains. We prove stability theorems and apply our results to two canonical models in recursive economics, generalising some known stability results to the cases when driving sequences are not independent and identically distributed. We also revisit the case of monotone Markovian models (or, equivalently, stochastic recursions with i.i.d. drivers) and provide a simplified version of the proof of a stability result given previously by Dubins and Freedman (1966) and Bhattacharya and Majumdar (2007).

Keywords: Monotone Economy, Markov Chain, Stochastic Recursion, Driving Sequence, Renegerative Sequence, Existence and Uniqueness of a Stationary Distribution, Stochastic Stability, Bewley-Huggett-Aiyagari Model, Risk-Sharing Model.

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1. INTRODUCTION

The dynamic evolution of a number of economic models can be described by a *stochastic recursive sequence (SRS)*, or *stochastic recursion* of the form (see, e.g., Stachurski, 2009, ch.6)

$$X_{t+1} = f(X_t, \xi_t) \quad \text{a.s.,} \tag{1}$$

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where $\{\xi_t\}$ is a stochastic process with $\xi_t \in \mathcal{S}$, $X \in \mathcal{X}$ is the state variable of economic interest and $f\colon \mathcal{X} \times \mathcal{S} \to \mathcal{X}$ is an appropriately measurable function. The process $\{\xi_t\}$ is known as the *driving sequence* of the stochastic recursion. For a given X_0 and given (random) values of $\xi_0, \ldots, \xi_{t-1}$, the system (1) generates a (random) value of X_t.

It is well-known (see, e.g., Borovkov and Foss, 1992) that, under extremely general conditions (see Section 2.1 for details), any time-homogeneous Markov chain may be represented as an SRS (1) with independent and identically distributed (i.i.d.) driving elements $\xi_0, \xi_1, \ldots$.

The questions we address in this paper are whether and under what conditions there exists an equilibrium distribution for X and if so, whether it is unique; whether the sequence X_t converges and if it does, whether the long-run distribution is independent of the initial conditions.

The answers to these questions depend on the spaces $\mathcal{X}$ and $\mathcal{S}$, the function f and the nature of the driving sequence. In this paper we are concerned with the case where the function f is monotone *increasing* and ξ is a *regenerative process*. We make appropriate assumptions on $\mathcal{X}$ and $\mathcal{S}$ to be specified below. Loosely, a stochastic process is regenerative if it can be split into independent and identically distributed (i.i.d.) cycles. That is, if there exists a subsequence of (random) dates such that the process has the same probabilistic behaviour between any two consecutive dates in the subsequence. The cycle lengths (lengths of time intervals between these dates) may also be random, in general, with the only requirement that they have a finite mean value. As an example, consider a finite-state time-homogeneous Markov chain with a single closed class of communicating states. If the chain starts in some state z_0, then the subsequence of dates corresponds to the dates at which the chain revisits state z_0. Between each of these dates the chain has the same probabilistic behaviour.[1] We present later a number of

[1] An i.i.d. process is one that is regenerative at every date.

further examples of processes that have such regenerative property. It allows us to obtain more general results on stability of monotone stochastic sequences, with driving sequences which are not necessarily i.i.d. In particular, we generalise the approach used by Bhattacharya and Majumdar (2007) that uses a *splitting condition* to establish stability.

If $\{\xi_t\}$ is i.i.d., then the process for X_t is Markov and standard existence and convergence results for Markov processes can be applied. For example, when f is monotone in the first argument, it is well-known that there is convergence to a unique invariant distribution if a mixing or splitting condition holds (see, e.g., Bhattacharya and Majumdar, 2007; Hopenhayn and Prescott, 1992; Stokey et al., 1989)[2].

Stability results are also known in a more general setting where the driving sequence $\{\xi_t\}$ is stationary or even asymptotically stationary (this literature originated with Loynes (1962), see, e.g., Borovkov and Foss (1992) and references therein). By *stationarity* we mean *stationarity in the strong sense*, that is, for any finite k, the distribution of a finite-dimensional vector $(\xi_t, \dots, \xi_{t+k})$ does not depend on t. The most basic result is that if the state space for the X's is partially ordered and possesses least element, say 0, and if SRS $X_{t+1} = f(X_t, \xi_t)$ starts from the bottom point $X_0 = 0$, with f monotone increasing in the first argument, then the distribution of X_t is monotone increasing in t and, given that the sequence is *tight*,[3] it converges to a limit which is the *minimal* stationary solution to recursion (1.1). In general, there may be many solutions, and for the minimal solution to be unique, one has to require existence of so-called *renovating events* (for de-

[2]Stokey et al. (1989) use the Feller property, which is a continuity requirement, together with monotonicity and a mixing condition to derive the results. Hopenhayn and Prescott (1992) develop an existence result using monotonicity alone, and combined with a mixing condition, establish that uniqueness and stability follow.

[3]Tightness in this context means that for any $\varepsilon > 0$ there exists K_ε such that $\mathbf{P}(X_t \geq K_\varepsilon) \leq \varepsilon$ for all t.

tails see, e.g., Brandt, 1985; Foss, 1983). These results seem to have been relatively little used in the economics literature although in Bewley (1987) it is assumed that there is a Markov driving sequence for shocks that starts from a stationary state.

It has long been recognised that since the interpretation of the driving process in many economic models is as an exogenous economic environment, the i.i.d. assumption is very restrictive and therefore, introducing a more general driving sequence, such as a driving sequence that is itself a Markov chain, is an important goal. For example, Donaldson and Mehra (1983) generalise the optimal stochastic growth model of Brock and Mirman (1972) to allow for positively correlated shocks to productivity. If $\{\xi_t\}$ is not i.i.d. but is determined by a "stationary transition function" (which, in our terminology, means that $\{\xi_t\}$ is a Markov chain itself, or SRS of the form $\xi_t = g(\xi_{t-1}, \varepsilon_{t-1})$ with i.i.d. $\{\varepsilon_t\}$), then the process $Y_t := (X_t, \xi_t)$ is clearly Markovian (see, e.g., Stokey et al. (1989, Chapter 9) or Hopenhayn and Prescott (1992, Corollary 5)). More precisely, if $\{\varepsilon_t\}$ is an i.i.d. sequence, then the pairs $Y_t = (X_t, \xi_t)$ represent a time-homogeneous Markov chain or, equivalently, an SRS of the form $Y_{t+1} = F(Y_t, \varepsilon_t) := (f(X_t, \xi_t), g(\xi_t, \varepsilon_t))$. Then one can use the monotone convergence approach to the study of the extended Markov chain Y_t. There are several disadvantages of this approach. First, in order to apply monotone convergence results we have to require that function f is monotone (increasing) in both arguments, and also that the transition function g for ξ_t is increasing in the first argument (while the latter restriction seems to be natural in some economic applications, the requirement for f to be increasing in the second argument is much more problematic). Second, the fact that the state space for the extended state variable is of a larger dimension, may create technical difficulties.[4]

We avoid these disadvantages by assuming a generalised version of the

[4]See Huggett (1993, Theorem 2).

splitting condition of Bhattacharya and Majumdar (2007) and monotonicity of f and allowing a much more general structure of ξ_t, assuming it to be *regenerative* (more precisely, we assume that the family of the ξ's is *driven* by a regenerative process; see Section 2 for details). Our first result, Theorem 1, reproduces a result of Bhattacharya and Majumdar (2007) for the i.i.d. case. The result is then generalized in Theorem 2 to allow for a regenerative driving sequence. A corollary to this theorem (Corolloary 1) gives the result in the important special case where the driving sequence is itself an aperiodic Markov chain with a positive atom.

We develop our approach in a simple scenario with a compact and completely ordered state space (which may be taken to be $[0,1]$ without loss of generality) and assuming an analogue of the monotone mixing condition of Bhattacharya and Majumdar (see, e.g., Bhattacharya and Majumdar, 1999, condition (1.2)) to hold. In the case of an i.i.d. driving sequence (we focus on this case for simplicity of notation and explanations; similar assumptions are needed in the Markovian case), this condition says that, for some $c \in [0,1]$, there is a finite time N such that, first, for the Markov chain $X_t^{(1)}$ that starts from the maximal state $X_0^{(1)} = 1$ at time zero (with any ξ_0), the probability of being below c after N jumps is positive, $\mathbf{P}(X_N^{(1)} \leq c) > 0$ and, second, for the Markov chain $X_t^{(0)}$ that starts from the minimal state $X_0^{(0)} = 0$ at time zero, the probability of being above c after N jumps is also positive, $\mathbf{P}(X_N^{(0)} \geq c) > 0$.

In many economic applications the function f is the policy function of a stochastic dynamic programming problem and a number of workhorse models have this structure. Examples include the optimal stochastic growth model of Brock and Mirman (1972), the Bewley-Imrohoroglu-Huggett-Aiyagari precautionary savings model of Bewley (1987), Imrohoroglu (1992), Huggett (1993) and Aiyagari (1994), and the risk-sharing under limited commitment model of Kocherlakota (1996) and Ligon et al. (2002). We apply our results to these cases and demonstrate that the convergence properties of the latter two models hold under less restrictive assumptions on the driving sequence than used in the current literature. This requires us to show that the policy functions and splitting condition are preserved with a more general driving sequence. In particular, we generalise the policy function results of Huggett (1993), who assumed that the driving process is a positively correlated (two-state) Markov chain to allow for any correlation structure.

The paper is organised as follows. In Section 2 we describe the model and provide our main results. First, we describe regenerative processes and give a number of examples. Next, we review the results of Bhattacharya and Majumdar (2007) for an i.i.d. driving sequence. Then, we present the main results showing that if a mixing condition similar to that given in Bhattacharya and Majumdar (2007) are satisfied between the dates when the driving sequence regenerates, then stability holds. Section 3 gives two applications of our main result. The proofs of the main result is given in the text and subsidiary proofs are put in the Appendix.

2. THE MAIN MODEL

In this section, we outline the main properties of regenerative processes, provide examples of regeneration for Markov chains, and finally introduce our main model, which is a stochastic recursive sequence with a regenerative driver.

Let $Z_n, n \geq 0$ be a (one-sided) regenerative sequence on a general measurable space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$. The sequence is *regenerative* if there exists an increasing sequence of integer-valued random variables (times) $0 = T_{-1} \leq T_0 < T_1 < T_2 < \ldots$ such that, for $\tau_n = T_n - T_{n-1}, n \geq 0$, the vectors

$$(2) \qquad \{\tau_n, Z_{T_{n-1}}, \ldots, Z_{T_n - 1}\}$$

are independent for $n \geq 0$ and identically distributed for $n \geq 1$. A random vector (2) is called a *cycle* with *cycle length* τ_n.

In this paper, we deal with discrete time and, therefore, assume T_n to be integer-valued. We further assume that

(3) $\mathbf{E}\tau_1 < \infty.$

It is known (see, e.g., Asmussen, 2003) that if, in addition, regenerative times are *aperiodic*,

(4) $G.C.D.\{n : \ \mathbf{P}(\tau_1 = n) > 0\} = 1,$

then Z_n has a unique stationary/limiting distribution, say π, and converges to it in the total variation norm:

$$\sup_{B \in \mathcal{B}_{\mathcal{Z}}} |\mathbf{P}(Z_n \in B) - \pi(B)| \to 0, \quad \text{as} \quad n \to \infty.$$

Remark 1 A typical regenerative scenario is produced by a positive recurrent time-homogeneous Markov chain with an atom (*Markov chain with a positive atom*, for short). This is a Markov chain with a general state space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ that contains a point $z_0 \in \mathcal{Z}$ such that, for any $z \in \mathcal{Z}$,

$$T_1^z = \min\{n \ : Z_n = z_0 \mid Z_0 = z\} < \infty \quad \text{a.s.}$$

and

$$\mathbf{E} T_1^{z_0} < \infty.$$

Below we give some examples of renegerative structures in the particular case where a Markov chain $\{Z_n\}$ takes values in a finite state space $\mathcal{Z}$ and is irreducible and aperiodic. Recall that, in this case, the Markov chain is positive recurrent, has a unique stationary distribution, say π, and, for any initial value $Z_0 = z$, there is convergence to π in the total variation norm. In the case of finite $\mathcal{Z}$, the sigma-algebra $\mathcal{B}_{\mathcal{Z}}$ contains all subsets of $\mathcal{Z}$.

Example 1 Pick any point $z_0 \in \mathcal{Z}$ and let

$$T_0 = \min\{n > 0 : Z_n = z_0\}$$

be the first time the process hits state z_0. Define further

$$T_j = \min\{n > T_{j-1} : Z_n = z_0\}$$

for all $j \geq 1$ to be the j-th time the process hits state z_0. Clearly the process $\{Z_n\}$ with the times $\{T_j\}$ possesses the regenerative structure.

EXAMPLE 2 Let $p(z_i, z_j) = \mathbf{P}(Z_1 = z_j | Z_0 = z_i)$ be the transition probabilities of the Markov chain. Take $K \geq 2$ and consider a sequence $(\omega_1, \omega_2, .., \omega_K) \in \mathcal{Z}^K$. If $p(\omega_i, \omega_{i+1}) > 0$ for each $i = 1, .., K-1$, then we say that this sequence forms a *word*. Let

$$T_0 \equiv T_0(\omega_1, \omega_2, .., \omega_K) = \min\{n \geq K : Z_{n-K+1} = \omega_1, Z_{n-K+2} = \omega_2, ..Z_n = \omega_K\}$$

be the first time the word $(\omega_1, \omega_2, .., \omega_K)$ appears in the sequence $\{Z_n\}$. Similarly, for each $j = 1, 2, \ldots$, let

$$T_j = \min\{n > T_j + K : Z_{n-K+1} = \omega_1, Z_{n-K+2} = \omega_2, ..Z_n = \omega_K\}$$

be the second, third, etc. times the same word appears in the sequence. Clearly, $\{Z_n\}$ is a regenerative process with regenerative times $\{T_j\}$.

EXAMPLE 3 In the setting of example 2, let, more generally, $k \in \{0, 1, 2, \ldots, K\}$ be any number. Let further $\widetilde{T}_j = T_j - k$, for $j = 0, 1, 2, \ldots$. Then again $\{Z_n\}$ is a regenerative process with regenerative times $\{\widetilde{T}_j\}$.

EXAMPLE 4 In the setting of the previous example 2, let $N \geq 1$ be any integer and let $\widehat{T}_j = T_{Nj}$. Then $\{Z_n\}$ is again a regenerative process with regenerative times $\{\widetilde{T}_j\}$.

EXAMPLE 5 Now we extend example 2 in another direction. Suppose there are $I \geq 2$ words $(\omega_{i,1}, \omega_{i,2}, .., \omega_{i,K_i})$ with corresponding lengths K_i, $i = 1, .., I$, such that $\omega_{1,K_1} = \omega_{2,K_2} = ... = \omega_{I,K_I}$ (i.e. they all end with the same character). Assume that none of these words is a "subword" of any other (i.e. cannot be obtained from another word by removing a number of characters at the beginning and/or at the end). Let

$$T_0 = \min_{i=1,..I} \{T_0(\omega_{i,1}, \omega_{i,2}, .., \omega_{i,K_i})\},$$

where $T_0(\omega_{i,1}, \omega_{i,2}, .., \omega_{i,K_i})$ are defined in example 2. In other words, here T_0 is the first appearance time of any of the words.

Let $K = \max_{i=1,\dots,I} K_i$ and define times $T_1, T_2, \dots$ by induction: given T_j, we let

$$T_{j+1} = \min\{n \geq T_j + K : (Z_{n-K_i+1}, \dots, Z_n) = (\omega_{i,1}, \dots, \omega_{i,K_i}) \quad \text{for some} \quad i = 1, \dots, I\}.$$

Then again $\{T_j\}$ forms a sequence of regenerative times for Markov chain $\{Z_n\}$.

The following example, which is a particular case of example 5, maybe considered as a simple example with *two states of the economy.*

Example 6 Consider a particular case of the previous example. Let the Markov chain have a two-point state space $\{1, 2\}$ and assume that $p(i, j) > 0$ for all $i, j \in \{1, 2\}$. We can then take any $K_1, K_2 \geq 3$ and assume the two words to be as follows:

$$\omega_{1,1} = .. = \omega_{1,K_1} = 1$$

and

$$\omega_{2,1} = .. = \omega_{2,K_2-2} = 2, \omega_{2,K_2-1} = \omega_{2,K_2} = 1.$$

Define times $\{T_j\}$ as in the previous example. These times are clearly regenerative for the process $\{Z_n\}$. Note also that the same applies to times $\{T_j - 1\}$ and $\{T_j - 2\}$.

Now we turn to the general framework. Assume that, along with regenerative process $\{Z_n\}$, we are given a family of random variables $\{\xi_n^z\}_{z\in\mathcal{Z},-\infty<n<\infty}$ that take values in a measurable space $(\mathcal{V},\mathcal{B}_{\mathcal{V}})$. We assume that

- this family does not depend on $\{Z_n\}$,
- contains mutually independent random variables,
- for each $z \in \mathcal{Z}$, random variables $\{\xi_n^z\}_{n\geq 1}$ are i.i.d. with common distribution G_z, and that,
- for each y, the function $G_z(y)$ is measurable with respect to z.

The main aim of the paper is to study the behaviour of a recursive sequence

$$(5) \qquad X_{n+1} = f\left(X_n, \xi_n^{Z_n}\right)$$

assuming that

- the function f is measurable and *monotone* in the first argument, with respect to some ordering;
- sequence $\{Z_n\}$ is regenerative and satisfies conditions (3)-(4).

2.1. *I.i.d. driving sequence*

We start with a particular case when $Z_n = z = const$ for each n. We then drop the upper index and simply write

$$(6) \qquad X_{n+1} = f\left(X_n, \xi_n\right),$$

where the ξ's are i.i.d. We revisit some results from Bhattacharya and Majumdar (2007) (see also Dubins and Freedman, 1966).

Recall that the relation between time-homogeneous Markov chains (with a general measurable state space $(\mathcal{X},\mathcal{B}_{\mathcal{X}})$ and recursions (6) with i.i.d. drivers is well-understood (see, e.g., Borovkov and Foss, 1992; Kifer, 1986) that if the sigma-algebra $\mathcal{B}_{\mathcal{X}}$ is countably generated, then a Markov chain may be represented as a stochastic recursion (6) with an i.i.d. driving sequence $\{\xi_n\}$. In particular, any real-valued or vector-valued time-homogeneous Markov chain may be represented as a stochastic recursion (6).

In what follows, we restrict our attention to real-valued X_n and, moreover, assume that

(7) the state space $\mathcal{X}$ is the closed interval $[0,1]$.

Introduce the *uniform distance* between probability distributions on the real line as

(8) $$d(F,G) = \sup_x |F(x) - G(x)|.$$

Here $F(x) = F(-\infty, x]$ and $G(x) = G(-\infty, x]$ are the distribution functions. Let $F(x-) = F(-\infty, x)$ and $G(x-) = G(-\infty, x)$. Then, clearly,

(9)
$$d(F,G) = \sup_x |F(x-) - G(x-)| \equiv \sup_x \max\left(|F(x-) - G(x-)|, |F(x) - G(x)|\right)$$

Next, we assume the function f to be *monotone increasing* in the first argument: for each $v \in \mathcal{V}$ and for each $0 \le x_1 \le x_2 \le 1$,

$$f(x_1, v) \le f(x_2, v).$$

Then, in particular, for any $v \in \mathcal{V}$ and any $y \in [0,1]$, the set

$$S(v,y) := \{x \ : \ f(x,v) \le y\}$$

is an interval containing 0 (i.e. interval of the form $[0,a)$ or $[0,a]$. In particular, it is a closed interval if f is continuous in x.

Similarly, the sets

$$S^{(2)}(v_1, v_2, y) := \{x \ : \ f(f(x, v_1), v_2) \le y\}$$

also form intervals containing zero, for all $v_1, v_2 \in \mathcal{V}$ and for all $y \in [0,1]$. Further, by the induction argument, we define similarly sets $S^{(n)}(\overline{v}, y)$ for all $y \in [0,1]$ and for all vectors $\overline{v} = (v_1, .., v_n)$ with $v_i \in \mathcal{V}$ for all i.

We write for short

$$\mathbf{P}^{(x)}(\cdot) = \mathbf{P}(\cdot \mid X_0 = x).$$

We also denote by $F_n^{(x)}$ the distribution function of the random variable X_n if $X_0 = x$. More generally, we denote by $F_n^{(\mu)}$ the distribution function of X_n if X_0 has distribution μ.

Theorem 1 *(see Bhattacharya and Majumdar, 2007, Theorem 3.5.1 for a slightly more general version).*
Assume that time-homogeneous Markov chain X_n is represented by the stochastic recursion (6) *with i.i.d. driving sequence $\{\xi_n\}$, where function $f : [0,1] \times \mathcal{V} \to [0,1]$ is monotone increasing in the first argument.*
Assume there exists a number $c \in [0,1]$ and integer $N \geq 1$ such that

$$\varepsilon_1 := \mathbf{P}^{(1)}(X_N \leq c) > 0$$

and

$$\varepsilon_2 := \mathbf{P}^{(0)}(X_N \geq c) > 0.$$

Then, there exists a distribution π on $[0,1]$ such that

$$\sup_x d(F_n^{(x)}, \pi) \to 0, \quad n \to \infty \tag{10}$$

exponentially fast.
Further, π is the unique stationary distribution for the Markov chain X_n.

In order to make the paper self-contained a concise proof of Theorem 1 that is useful for proving our main result (Theorem 2) is given in the Appendix. The first results in this direction were obtained in Dubins and Freedman (1966) (under an additional assumption of continuity of the mapping f).

Remark 2 Theorem 1 is easily generalized to a case where the set $\mathcal{S}$ has a partial order, $\leq$, such that there exists least element $s_0 \in \mathcal{S}$ and greatest element $s_1 \in \mathcal{S}$ and f is monotone increasing in the first argument (with respect to the partial order $\leq$). In this case, the mixing condition requires that there exists a $\varepsilon > 0$, an integer $N \geq 1$ and sets $\mathcal{C}_u \subset \mathcal{S}$ and $\mathcal{C}_l \subset \mathcal{S}$ such that for every element $s \in \mathcal{S}$, there either exists an element $c \in \mathcal{C}_u$ such that $s \geq c$, or there exists an element $c \in \mathcal{C}_l$ such that $s \leq c$; and for every $c \in \mathcal{C}_u$, $\mathbf{P}^{(s_1)}(X_N \leq c) > \varepsilon$, and for every $c \in \mathcal{C}_l$, $\mathbf{P}^{(s_0)}(X_N \geq c) > \varepsilon$.

Remark 3 Note that the assumptions of Theorem 1 may be rewritten in an equivalent form as follows: assuming in addition that the trajectories $X_t^{(1)}$ and $X_t^{(0)}$ are mutually independent, there is a positive N such that $\mathbf{P}(X_N^{(1)} \leq X_N^{(0)}) \geq \delta > 0$. This condition is called a *strong reversing condition* by Kamihigashi and Stachurski (2014) because then, due to the monotonicity, it also holds for any other pair of initial conditions $0 \leq x_0 < y_0 \leq 1$, with the same δ and N, namely $\mathbf{P}(X_N^{(y_0)} \leq X_N^{(x_0)}) \geq \delta > 0$. One can consider our approach in a more general setting (developed for Markov chains by Kamihigashi and Stachurski (2014) and Szeidl (2013)), assuming, more generally, that the state space may not contain top and/or bottom points (then the δ and N may, in general, depend on (x_0, y_0)) and, moreover, that the order is only partial. In particular, Szeidl (2013) suggested a reasonable "replacement", say, for a top point (if one does not exist) by a random "top" point. Namely, assume, say, the state space for the Markov chain is the positive half-line $[0, \infty)$ where there is the minimal element 0 but there is no maximal element, and suppose that a Markov chain X_t is defined by a stochastic recursion $X_{t+1} = f(X_t, \xi_t)$ with i.i.d. $\{\xi_t\}$. Assume that there exists a random measure μ on $[0, \infty)$ such that if $X_0 \sim \mu$ and if X_0 does not depend on ξ_0, then $X_1 = f(X_0, \xi_0)$ is *stochastically smaller* than X_0 (this means $\mathbf{P}(X_1 \leq x) \geq \mathbf{P}(X_0 \leq x)$, for all x). Then the distribution μ may play a role of a new random "top" point if, for example, the distribution of μ has an unbounded support. For instance, if there exists another function, say h such that $f(x,y) \le h(x,y)$ for all x,y and that a Markov chain $Y_{t+1} = h(Y_t, \xi_t)$ admits a unique stationary distribution, say μ. If μ can be easily found/determined, it may play the role of a random "top" point.

Here is a simple example. Assume that X_t is a discrete-time birth-and-death-process with immigration at 0, i.e. a non-negative integer-valued Markov chain, which is homogeneous in time and with transition probabilities $\mathbf{P}(X_1 = 1 \mid X_0 = 0) = 1 - \mathbf{P}(X_1 = 0 \mid X_0 = 0) = p_0 > 0$ and, for $k = 1, 2, \ldots$, let $\mathbf{P}(X_1 = k+1 \mid X_0 = k) = 1 - \mathbf{P}(X_1 = k-1 \mid X_0 = k) = p_k$. Assume further that the p_k are non-decreasing in k (this makes Markov chain monotone), that all are smaller than $1/2$ and, moreover, that $\lim_{k\to\infty} p_k = p < 1/2$. Consider a Markov chain Y_t with transition probabilities $\mathbf{P}(Y_1 = k+1 \mid Y_0 = k)) = p = 1 - \mathbf{P}(Y_1 = \max(0, k-1)) \mid Y_0 = k)$. Then this Markov chain has a unique stationary distribution μ (which is clearly geometric), and it gives a random "top" point.

In Borovkov and Foss (1992), a similar concept of a stationary majorant was developed and studied, where the top sequence $\{X_n\}$ is assumed to be stationary.

Similar ideas have been developed earlier in the area of so-called "perfect simulation", with introducing an artificial random "top" point (see, e.g., Corcoran and Tweedie, 2001, and the references therein) for simulation "from the past".

2.2. *Regenerative driving process*

We now turn our attention to the general regenerative setting (5), but continue to assume (7) to hold. We formulate and prove a general result and then give some important corollaries and examples.

We introduce an auxiliary process $\widetilde{X}_n^{(a)}$ that starts from $\widetilde{X}_0^{(a)} = a$ at time

0, and follows the recursion

$$\widetilde{X}^{(a)}_{n+1} = f\left(\widetilde{X}^{(a)}_n, \xi_n^{Z_{T_0+n}}\right) \quad \text{for all} \quad n \geq 0.$$

REMARK 4 The auxiliary process $\widetilde{X}^{(a)}_n$ coincides in distribution with the process X started at time T_0 from the state a, and assumptions (11) and (12) ensure the mixing (similar to that guaranteed by conditions of Theorem 1) over a typical cycle (from T_0 to T_1) of the regenerative process Z.

More generally, we consider an auxiliary process $\widetilde{X}^{(F)}_n$ that follows the recursion

$$\widetilde{X}^{(F)}_{n+1} = f\left(\widetilde{X}^{(F)}_n, \xi_n^{Z_{T_0+n}}\right) \quad \text{for all} \quad n \geq 0$$

and that starts from a random variable $\widetilde{X}^{(F)}_0$ that has distribution F (and which does not depend on random variables $\{Z_{T_0+n}, \xi^{(z)}_n, z \in \mathcal{Z}, n \geq 0\}$). Denote by $f^{(k)}$ the k-th iteration of function f, so $f^{(1)} = f$ and, for, $k \geq 1$,

$$f^{(k+1)}(x, u_1, \ldots, u_{k+1}) = f\left(f^{(k)}(x, u_1, \ldots, u_k), u_{k+1}\right).$$

Let $f^{(0)}$ be the identity function.

THEOREM 2 *Assume that recursive sequence* $\{X_n\}$ *is defined by* (5) *where the function* f *is monotone increasing in the first argument and the sequence* $\{Z_n\}$ *is regenerative with regenerative times* $\{T_n\}$ *that satisfy conditions* (3)-(4).

Assume that the following assumptions hold:

(11) $$\varepsilon_1 := \mathbf{P}\left(\widetilde{X}^{(1)}_{T_1-T_0} \leq c\right) > 0,$$

and

(12) $$\varepsilon_2 := \mathbf{P}\left(\widetilde{X}^{(0)}_{T_1-T_0} \geq c\right) > 0.$$

Then there exists a distribution π *on* $[0,1]$ *such that*

(13) $$\rho_n := \sup_x d(G^{(x)}_n, \pi) = \sup_x \sup_t |G^{(x)}_n(t) - \pi(-\infty, t]| \to 0, \quad n \to \infty$$

exponentially fast. Here $G_n^{(x)}$ *is the distribution of* X_{T_n} *if* $X_{T_0} = x$.

Further, the distributions of X_n *converge to distribution*

$$\mu(\cdot) = \frac{1}{\mathbf{E}(\tau_1)} \sum_{k=0}^{\infty} \mathbf{P}\left(\tau_1 > k, f^{(k)}\left(\widetilde{X}_0^{(\pi)}, \xi_0^{Z_{T_0}}, \ldots, \xi_{k-1}^{Z_{T_0+k-1}}\right) \in \cdot\right),$$

for any initial value X_0*, again in the uniform metric* d.

Remark 5 Note that, as in the Markovian case of Theorem 1, we do not require that the function f be continuous in the first argument.

Proof: Introduce a sequence $Y_{n-1} = X_{T_n}$ for all $n \geq 0$. This sequence is clearly a Markov chain and can therefore be represented in the form (6)

$$Y_{n+1} = g(Y_n, \eta_n)$$

with an i.i.d. driving sequence

$$\eta_n = \left(\tau_n, \xi_{T_{n-1}}^{Z_{T_{n-1}}}, .., \xi_{T_n-1}^{Z_{T_n-1}}\right)$$

and the function g is defined by

$$g(Y_n, \eta_n) = f^{(\tau_n)}\left(Y_n, \xi_{T_{n-1}}^{Z_{T_{n-1}}}, .., \xi_{T_n-1}^{Z_{T_n-1}}\right).$$

In addition, this recursion is again monotone in the first argument, due to the monotonicity of function f.

The assumptions of the theorem imply that there exists $c \in [0,1]$ such that

$$\mathbf{P}(Y_1 \leq c | Y_0 = 1) = \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(1)} \leq c\right) > 0$$

and

$$\mathbf{P}(Y_1 \geq c | Y_0 = 0) = \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(0)} \geq c\right) > 0.$$

Hence, the assumptions of Theorem 1 are satisfied with the same c and with $N = 1$. This implies the first statement of the theorem.

We prove the second statement now.

For any n, let $\nu(n)$ be such that $T_{\nu(n)} \le n < T_{\nu(n)+1}$. Let $\psi_n = (n - \nu(n), \xi_{\nu(n)}^{Z_{\nu(n)}}, \ldots, \xi_{n-1}^{Z_{n-1}})$ and denote $\psi_{n,1} = n-\nu(n)$ and $\psi_{n,2} = (\xi_{\nu(n)}^{Z_{\nu(n)}}, \ldots, \xi_{n-1}^{Z_{n-1}})$, so $\psi_n = (\psi_{n,1}, \psi_{n,2})$.

By the classical result on regenerative processes (see e.g., Asmussen (2003)), for any fixed $k > 0$ and as n tends to infinity, the joint distribution of random vectors $(\eta_{\nu(n)-k}, \eta_{\nu(n)-k+1}, \ldots, \eta_{\nu(n)}, \psi_n)$ converges in the total variation norm D to the limiting distribution of, say, random vectors $(\eta^{-k}, \ldots, \eta^0, \psi^0)$ that are mutually independent:

$$D_{n,k} := \sup_B |\mathbf{P}((\eta_{\nu(n)-k}, \ldots, \eta_{\nu(n)}, \psi_n) \in B) - \mathbf{P}((\eta^{-k}, \ldots, \eta^0, \psi^0) \in B)| \to 0.$$

In addition, each of the η^{-j}, $j = 0, \ldots, k$, has the distribution of the "typical cycle", while random vector ψ^0 represents the left half of the "integrated cycle", and its first coordinate ψ_1^0 has the integrated tail distribution $\mathbf{P}(\psi_1^0 = l) = \frac{1}{\mathbf{E}\tau_1}\mathbf{P}(\tau_1 > l)$, for $l = 0, 1, \ldots$. In what follows, we use representation $\psi^0 = (\psi_1^0, \psi_2^0)$ where ψ_2^0 is the rest of vector ψ^0 (and, in particular, it is l-dimensional if $\psi_1^0 = l$).

Further, for any fixed k and n, there is a coupling of $(\eta_{\nu(n)-k}, \ldots, \eta_{\nu(n)}, \psi_n)$ and of $(\eta^{-k}, \ldots, \eta^0, \psi^0)$ such that

$$\mathbf{P}((\eta_{\nu(n)-k}, \ldots, \eta_{\nu(n)}, \psi_n) \neq (\eta^{-k}, \ldots, \eta^0, \psi^0)) = D_{n,k}$$

(see, e.g. (Lindvall, 2002, Chapter 1)). In the rest of the proof, we consider a $(\eta^{-k}, \ldots, \eta^0, \psi^0)$ where there is such a coupling.

Introduce $\widehat{Y}_{n,0} = Y_{\nu(n)-k}$ and

$$\widehat{Y}_{n,m+1} = g(\widehat{Y}_{n,m}, \eta_{\nu(n)-k+m}), \quad m = 0, .., k-1$$

and

$$\widetilde{Y}_{n,m+1} = g(\widehat{Y}_{n,m}, \eta^{\nu(n)-k+m}), \quad m = 0, .., k-1.$$

Consider now

$$
\begin{aligned}
&|\mathbf{P}(\widehat{Y}_{n,k} \le t) - \mathbf{P}(\widetilde{Y}_{n,k} \le t)| \\
&= |\mathbf{P}(g^{(k)}(\widehat{Y}_{n,0}, (\eta_{\nu(n)-k}, \dots, \eta_{\nu(n)})) \le t) - \mathbf{P}(g^{(k)}(\widehat{Y}_{n,0}, (\eta^{-k}, \dots, \eta^{0})) \le t)| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le D_{n,k}
\end{aligned}
$$

for any t, with the obvious notation for $g^{(k)}$. This means that the distributions of $\widehat{Y}_{n,k}$ and $\widetilde{Y}_{n,k}$ are close in the uniform metric d. Due to the first statement of the theorem, we also have that, as $k \to \infty$, the distribution of the random variable $\widetilde{Y}_{n,k}$ converges to distribution π in the total variation norm and, hence, in the uniform metric. We can therefore, for any $\varepsilon > 0$, choose k such that, for any t,

$$|\mathbf{P}(\widehat{Y}_{n,k} \le t) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le t)| \le \varepsilon$$

and

$$|\mathbf{P}(\widetilde{Y}_{n,k} \le t) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le t)| \le D_{n,k} + \varepsilon.$$

Now, for any t and any $l = 0, 1, \dots$,

$$
\begin{aligned}
&|\mathbf{P}(X_n \le t, n - \nu(n) = l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le t, \psi_1^0 = l| \\
&= |\mathbf{P}(f^{(l)}(\widehat{Y}_{n,k}, \psi_{n,2}) \le t, n-\nu(n) = l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le t, \psi_1^0 = l| \\
&\le |\mathbf{P}(f^{(l)}(\widehat{Y}_{n,k}, \psi_{n,2}) \le t, n-\nu(n) = l) - \mathbf{P}(f^{(l)}(\widehat{Y}_{n,k}, \psi_2^0) \le t, \psi_1^0 = l)| \\
&\quad + |\mathbf{P}(f^{(l)}(\widehat{Y}_{n,k}, \psi_2^0) \le t, \psi_1^0 = l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le t, \psi_1^0 = l|.
\end{aligned}
$$

The first summand does not exceed $\mathbf{P}(n - \nu_n \ne \psi_0^1)$ and therefore converges to zero (uniformly in t). Now turn to the second summand. For any $l = 1, 2, \dots$ and any $v \in \mathcal{V}^l$, the set $S_l(v,t) = \{x \ : \ f^{(l)}(x,v) \le l\}$ is an interval of the form $[0,a)$ or $[0,a]$, for some a. Therefore,

$$
\begin{aligned}
& |\mathbf{P}(f^{(l)}(\widehat{Y}_{n,k}, v) \le t) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, v) \le t)| \\
= \ & |\mathbf{P}(\widehat{Y}_{n,k} \in S_l(v,t)) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \in S_l(v,l))| \\
\le \ & \sup_w |\mathbf{P}(\widehat{Y}_{n,k} \le w) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le w)|.
\end{aligned}
$$

Then the second summand in the earlier inequality is not bigger than

$$\begin{aligned}
& \mathbf{P}(\psi_1^0 = l) \int |\mathbf{P}(f^{(l)}(\widehat{Y}_{n,k}, v) \le t) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, v) \le t)| \mathbf{P}(\psi^0 \in dv \mid \psi_1^0 = l) \\
\le \; & \mathbf{P}(\psi_1^0 = l) |\mathbf{P}(\widehat{Y}_{n,k} \le w) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le w)|
\end{aligned}$$

Thus,

$$\sup_t |\mathbf{P}(X_n \le t, n - \nu(n) = l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le t, \psi_1^0 = l)|$$

tends to 0, and the same holds for any finite sum in l.

Since the family of distributions of random variables $n - \nu(n)$ is tight, the second statement of the theorem follows. *Q.E.D.*

Remark 6 In general, we require only the first moment of τ_1 to be finite, so convergence in the regeneration theorem may be as slow as one wishes, and the same holds for convergence of the distribution F_n of random variable X_n to μ. However, if τ_1 has finite $(1+r)$-th moment, then $d(F_n, \mu)$ decays not slower than n^{-r}; and if τ_1 has finite exponential moment, then the convergence is exponentially fast.

Example 7 Consider a very simple toy example, with only two states of environment $\mathcal{Z} = \{1, 2\}$ and with four-state space $\mathcal{X} = \{0, 1, 2, 3\}$ (we prefer to deal with integers, so we rescale $[0, 1]$ to $[0, 3]$). Assume sequence $\{Z_n\}$ to be regenerative, with the typical cycle to take two values, $(2, 1)$ and $(2, 2, 1)$ with equal probabilities $1/2$, so the cycle length τ_1 is either 2 or 3, with mean $\mathbf{E}\tau_1 = 5/2$. The stochastic recursion is given by

$$X_{n+1} = \min(3, \max(0, X_n + \xi_n^{Z_n}))$$

where $\mathbf{P}(\xi_n^1 = k) = 1/4$ for $k = 0, 1, 2, 3$ and $\mathbf{P}(\xi_n^2 = -1) = \mathbf{P}(\xi_n^2 = -2) = 1/2$. It may be easily checked that the SRS satisfies all the conditions of the previous theorem.

Introduce the embedded Markov chain $Y_n = X_{T_n}$, like in the proof of the previous theorem. It is irreducible with transition probability matrix $P = \{p_{i,j}, 0 \le i, j \le 3\}$ given by

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 3/16 & 1/4 & 1/4 & 5/16 \\ 3/32 & 3/16 & 1/4 & 15/32 \end{pmatrix}$$

For example, here

$$\begin{aligned} p_{3,1} &= \mathbf{P}(\tau_1 = 2, \xi_1^2 = -2, \xi_2^1 = 0) + \mathbf{P}(\tau_1 = 3, \xi_1^2 = \xi_2^2 = -1, \xi_3^1 = 0) \\ &+ \mathbf{P}(\tau_1 = 3, \xi_1^2 + \xi_2^2 = -3, \xi_3^1 = 1) + \mathbf{P}(\tau_1 = 3, \xi_1^2 = \xi_2^2 = -2, \xi_3^1 = 1) \\ &= \frac{1}{16} + \frac{1}{32} + \frac{1}{16} + \frac{1}{32} = \frac{3}{16}. \end{aligned}$$

Then the distribution of Y_n converges to $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ which may be found by solving $\pi P = \pi$ with $\sum \pi_i = 1$. So we get $\pi = (29/160, 183/800, 1/4, 17/50)$. Further, the limiting distribution for X_n is given by

$$\begin{aligned} \mu_k &= \frac{1}{\mathbf{E}\tau_1}(\mathbf{P}(Y^{(0)} = k) + \mathbf{P}(\max(0, Y^{(0)} + \xi_0^2) = k) \\ &+ \mathbf{P}(\max(0, Y^{(0)} + \xi_0^2 + \xi_1^2) = k, \tau_1 = 3)), \end{aligned}$$

for $k = 0, 1, 2, 3$, where $Y^{(0)} \sim \pi$. In particular, $\mu_3 = 2\pi_3/5$, $\mu_2 = \frac{2}{5}(\pi_2 + \pi_3/2)$, and $\mu_1 = \frac{2}{5}(\pi_1 + (\pi_2 + \pi_3)/2 + \pi_3/8) = \frac{2}{5}(\pi_1 + \pi_2/2 + 5\pi_3/8)$.

2.3. *The case where the governing sequence is Markov*

In the particular case where $\{Z_n\}$ is a Markov chain, Theorem 2 leads to the following corollary.

Corollary 1 *Assume again that the recursive sequence $\{X_n\}$ is defined by (5), and that the function f is monotone increasing in the first argument. Assume further that $\{Z_n\}$ is an aperiodic Markov chain with a positive atom at point z_0 (see remark before example 1). Assume also that there exists a number $0 \le c \le 1$, positive integers N_1 and N_2 and sequences $z_{1,1}, \ldots, z_{N_1,1}$*

and $z_{1,2}, \ldots, z_{N_2,2}$ *such that* $z_{N_1,1} = z_{N_2,2} = z_0$ *and, for* $i = 1, 2$, *the following hold:*

$$p_i := \mathbf{P}(Z_j = z_{j,i},\ \textit{for}\ \ j = 1, \ldots, N_i \mid Z_0 = z_0) > 0$$

and that

$$\delta_1 := \mathbf{P}(\widetilde{X}^{(1)}_{N_1} \le c \mid Z_0 = z_0,\ Z_j = z_{j,1},\ j = 1, \ldots, N_1) > 0$$

and

$$\delta_2 := \mathbf{P}(\widetilde{X}^{(0)}_{N_2} \ge c \mid Z_0 = z_0,\ Z_j = z_{j,2},\ j = 1, \ldots, N_2) > 0.$$

Then the distribution of X_n *converges in the uniform metric to a unique stationary distribution.*

Proof: We have to show that Corollary 1 follows from Theorem 2. For that, we have to define a typical (say, first) regenerative cycle and show that all the conditions of Theorem 2 hold.

Assume that $Z_0 = z_0$, so $T_0 = 0$. Let $T_1 = \tau_1 = \min\{n > 0 : Z_n = z_0\}$, then the aperiodicity means that $G.C.D.\{n : \mathbf{P}(T_1 = n) > 0\} = 1$. Let $T_n = \sum_1^n \tau_j$ where τ_j are i.i.d. copies of τ_1.

Let the conditions of the Corollary hold, and let k_i be the number of occurrences of z_0 in the sequence $z_{j,i}$, for $i = 1, 2$. Let L be the *least common multiple* of k_1 and k_2,

$$L = \min\{l\ :\ l/k_1\ \text{ and }\ l/k_2\ \text{ are integers}\}.$$

Let α be a random variable taking values 0 and 1 with equal probabilities and does not depend on any of the processes defined in the model. Then define a regenerative cycle as follows: $\widehat{T}_0 = 0$ and

$$\widehat{T}_1 = T_1\alpha + T_L(1 - \alpha).$$

That is, we suppose that our regenerative cycle is either a single cycle or a sum of L cycles, with equal probabilities. Then all the conditions of Theorem 2 hold (with $\widehat{T}_i$ in place of T_i). Indeed, condition (3) follows since it holds for τ_1, and since $\widehat{T}_1$ is not bigger than T_L, the sum of L copies of τ_1. Condition (4) follows because the set of all n such that $\mathbf{P}(\widehat{T}_1 = n) > 0$ includes the set of all n such that $\mathbf{P}(\tau_1 = n) > 0$ and, therefore,

$$G.C.D.\{n \ : \ \mathbf{P}(\widehat{T}_1 = n) > 0\} \leq G.C.D.\{n \ : \ \mathbf{P}(\tau_1 = n) > 0\},$$

so, given aperiodicity, both greatest common divisors are equal to 1. Finally, ε_1 in (11) is not smaller than $\frac{1}{2}p_1\delta_1 > 0$ and, similarly, ε_2 in (12) is not smaller than $\frac{1}{2}p_2\delta_2 > 0$. *Q.E.D.*

3. APPLICATIONS

In this section we present two workhorse models. We extend known stability results by weakening the assumption on the underlying driving process.

3.1. *Bewley-Huggett-Aiyagari Precautionary Savings Model*

The basic income fluctuation model in which many risk-averse agents self-insure against idiosyncratic income shocks through borrowing and saving using a risk-free asset, is dubbed by Heathcote et al. (2009) "the standard incomplete markets model" and is the workhorse model in quantitative macroeconomics. At its heart is the stochastic savings models of Huggett (1993) with an exogenous borrowing constraint, or close variants of this model.[5] As Heathcote et al. (2009) observe, there are "few general results that apply to this class of problems." Existing analytical work in the standard model requires, to the best of our knowledge, either that income fluctuations are i.i.d., or that an individual's income process is "persistent" or

[5]Bewley (1987) and Aiyagari (1994) vary the context but the individual savings problem is similar. They each derive existence and convergence results under slightly different assumptions. Bewley assumes that the endowment shocks are stationary Markov, Huggett (1993) assumes positive serial correlation and two states, and Aiyagari assumes that endowment shocks are i.i.d. Imrohoroglu (1992) uses numerical methods with a two state persistent income process as in Huggett.

monotone: a higher income today implies that income tomorrow is higher in the stochastic dominance sense. While the latter is a natural assumption to make in many contexts, it is nonetheless restrictive, and we show here that our results allow for an arbitrary stationary income process in Huggett's model with a finite number of states (we maintain all his other assumptions).[6] Obvious examples of non-monotone processes would include termination pay where a worker receives a one-off substantial payment on cessation of an employment, but is then unemployed, and health shocks where an insurance payout is received but future employment prospects are diminished.

Finding a steady state in these models requires identifying an interest rate R^*, say, so that at R^* there is a stationary distribution for individual wealth – reflecting different individual histories of income shocks – such that net asset demand is zero. Apart from existence, important ingredients of this approach in practice are: uniqueness of the invariant distribution at any given R, which is important for continuity of the asset excess demand function; and stability, which matters both as it implies a straightforward approach for computing net asset demand at any candidate R (weak convergence is sufficient), and also as a heuristic justification for focusing on the steady state.

Agents maximize expected discounted utility

$$\mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t u(c^t)\right],$$

where $c^t \in \mathbb{R}_+$ is consumption at time t, $t = 0, 1, \ldots$, $u(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma > 1$, subject to a budget constraint at each date

$$c + R^{-1}a' \leq a + e, \tag{14}$$

a borrowing constraint $a' \geq \underline{a}$, where e is the current endowment, a is current

[6]Miao (2002) extends Huggett's model from two states to many states.

assets, a' is assets next period, c is consumption, $\beta \in (0,1)$ is the discount factor and $R^{-1} > \beta$ is the price of next-period assets. It is assumed that $e \in \{\widehat{e}_1, \ldots, \widehat{e}_n\} =: E$, $n \geq 2$, $\widehat{e}_n > \widehat{e}_1$, $\widehat{e}_{i+1} \geq \widehat{e}_i > 0$, all $i = 1, \ldots, n-1$, and $\underline{a} < 0$ and $\underline{a} + \widehat{e}_1 - \underline{a}R^{-1} > 0$. The individual's endowment e^t is governed by a Markov chain with stationary transition probabilities $p(i,j) := \Pr(e^{t+1} = \widehat{e}_j \mid e^t = \widehat{e}_i) > 0$, $i,j = 1, 2, \ldots, n$. We assume initial values $e^0 \in E$ and $a^0 \geq \underline{a}$ are given.

The individual's decision problem can be represented by the functional equation:

$$v(a,e) = \max_{(c,a') \in \Gamma(a,e)} u(c) + \beta \mathrm{E}_{e'|e} v(a', e') \tag{15}$$

where

$$\Gamma(a,e) = \left\{ (c, a') \mid c + R^{-1}a' \leq a + e, a' \geq \underline{a}, c \geq 0 \right\}$$

is the constraint set and $v(a,e)$ are the value functions (one for each realisation e). The resulting policy functions are denoted $c = c(a,e)$ and $a' = f(a,e)$. Huggett (Theorem 1) proves that there is a unique, bounded and continuous solution to (15) and each $v(a,e)$ is increasing, strictly concave and continuously differentiable in a, while f is continuous and nondecreasing in a, and (strictly) increasing whenever $f(a,e) > \underline{a}$. These results extend to our context; see Miao (2002).

Huggett assumes $n = 2$ and persistence of the endowment process: $p(2,2) \geq p(1,2)$, and shows that for a given R, there exists a unique stationary probability measure for $x = (a,e)$ and that there is weak convergence to this distribution for any initial distribution on x (Huggett [Theorem 2]).

We can extend this result to our more general context using the following (the proof can be found in the Appendix).[7]

[7] Similar properties are established in Huggett (1993), and in Miao (2002) for the many state case, but using the persistence assumption.

Lemma 1 *(i)For $a > \underline{a}$, there is at least one $e \in E$ such that $f(a,e) < a$; (ii) there exists $\widehat{a} \geq \underline{a}$ such that for all $a > \widehat{a}$, all $e \in E$, $f(a,e) < a$.*

Define $\bar{a} := \min_a\{a \geq \underline{a} : f(a,e) \leq a \text{ all } e \in E\} < \infty$. (This exists by part (ii) of the lemma, given the continuity of $f(\cdot, e)$.) If $\bar{a} > \underline{a}$, then define $\underline{f}(a) := \min_e\{f(a,e)\}$, where $\underline{f}(a) < a$ on $(\underline{a}, \bar{a}]$ by part (i) of the lemma, while $\bar{f}(a) := \max_e\{f(a,e)\} > a$ on $[\underline{a}, \bar{a})$ by definition of $\bar{a}$. Starting from $(\bar{a},\, e_0)$, repeatedly applying $\underline{f}$ yields the strictly decreasing sequence $(\underline{f}(f(\bar{a}, e_0)), \underline{f}^{(2)}(f(\bar{a}, e_0)), \ldots)$ where $\underline{f}^{(n)}$ denotes the n-fold composition of $\underline{f}$. Suppose that $\lim_{T\to\infty} \underline{f}^{(T)}(f(\bar{a}, e_0)) = \widetilde{a} > \underline{a}$. Then by continuity of $\underline{f}$, $\underline{f}(\widetilde{a}) = \widetilde{a}$, which contradicts part (i) of the lemma. So $\lim_{T\to\infty} \underline{f}^{(T)}(f(\bar{a}, e_0)) = \underline{a}$, and fixing some $a' \in (\underline{a}, \bar{a})$, there exists a finite sequence $(e_t)_{t=1}^{T}$ with $e_t \in \arg\min_{e\in E}\{f(\underline{f}^{(t-1)}(f(\bar{a}, e_0)), e)\}$ such that the occurrence of $(e_t)_{t=1}^{T}$ implies $a_t < a'$. Moreover $(e_t)_{t=1}^{T}$ has positive probability. By a symmetric argument using $\bar{f}(a)$, starting from $(\underline{a}, e_0)$, there exists a positive probability, finite sequence of endowment shocks $(\widetilde{e}_t)_{t=1}^{\widetilde{T}}$ whose occurrence implies $a_t > a'$. Let $\{Z_n\}$ in Corollary 1 be identified with the process $\{e_n\}$, assuming that each $\xi_n^{Z_n}$ has a degenerate distribution at e_n, so that $f(X_n, \xi_n^{Z_n}) = f(a_n, e_n)$. Setting $z_0 = e_0$, $c = a'$, $z_{j,2} = e_j$, $N_2 = T - 1$, $z_{j,1} = \widetilde{e}_j$, $N_1 = \widetilde{T} - 1$, and given $p(i,j) > 0$ for each i and j, the conditions of the corollary are all satisfied.

This implies that there exists a unique distribution π such that the distributions of a_t converge to π in the uniform metric for any initial value $a_0 \in [\underline{a}, \bar{a}]$. (If $\bar{a} = \underline{a}$ then this would be a degenerate limit.) Moreover, any subset of $(\bar{a}, \infty)$ is transient.

3.2. *Limited Commitment Risk-Sharing Model*

In this section we consider the inter-temporal risk-sharing model with limited commitment. Kocherlakota (1996) provides a convergence result for the long-run distribution of risk-sharing transfers when shocks to income are finite and i.i.d. His model has two, infinitely-lived, risk averse agents with per-period, strictly concave and differentiable utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ defined over consumption, and a common discount factor β. Agent 1 has a random endowment $\xi_t > 0$ at date $t = 0, 1, \ldots,$ and agent 2 has a random endowment $Y - \xi_t > 0$ where $Y > 0$ is a constant aggregate income. There is no credit market but agents can transfer income between themselves at any date. Although Kocherlakota assumes the income hocks are i.i.d., we now show has convergence result is easily extended to the case where $\{\xi_t\}$ is a finite state Markov chain with stationary transition probabilities (as in the previous example). Letting $h^t = (\xi_0, \xi_1, \ldots, \xi_t)$ denote the history of income realisations, agents choose a sequence of history-dependent transfers $X_t(h^t)$ from agent 1 to agent 2 subject to $-Y + \xi_t \leq X_t(h^t) \leq \xi_t$ for each h^t and the self-enforcing constraints that neither agent prefers autarky from that point on after any history over the agreed transfer plan. In particular, the self-enforcing constraints for the two agents are

$$\begin{aligned} u(\xi_t - X_t(h^t)) \quad &+ \quad \mathrm{E}_t[\sum_{s=1}^{\infty} \beta^s u(\xi_{t+s} - X_t(h^{t+s}))] \\ &\geq \quad u(\xi_t) + \mathrm{E}_t[\sum_{s=1}^{\infty} \beta^s u(\xi_{t+s}))], \\ u(Y - \xi_t + X_t(h^t)) \quad &+ \quad \mathrm{E}_t[\sum_{s=1}^{\infty} \beta^s u(Y - \xi_{t+s} + X_t(h^{t+s}))] \\ &\geq \quad u(Y - \xi_t) + \mathrm{E}_t[\sum_{s=1}^{\infty} \beta^s u(Y - \xi_{t+s}))], \end{aligned}$$

for each date t and h^t. An *efficient risk-sharing arrangement* will solve (for some feasible U^0):

$$\max_{\{X_t\}} \mathrm{E}_0[\sum_{s=0}^{\infty} \beta^s u(\xi_s - X_s(h^s))] \quad \text{s.t.} \quad \mathrm{E}_0[\sum_{s=0}^{\infty} \beta^s u(Y - \xi_s + X_s(h^s))] \geq U^0.$$

and subject to the self-enforcing constraints. It is well known (see, e.g., Ligon et al., 2002) that the solution at each date has the following property: For each realisation ξ, there is an interval $I_\xi = [\underline{c}_\xi, \overline{c}_\xi]$, $\underline{c}_\xi \leq \overline{c}_\xi$, such that

$$c_{t+1}\left(h^{t+1}\right) := \xi_{t+1} - X_{t+1}(h^{t+1}) = \begin{cases} \overline{c}_{\xi_{t+1}} & \text{if } c_t(h^t) > \overline{c}_{\xi_{t+1}} \\ c_t(h^t) & \text{if } c_t(h^t) \in I_{\xi_{t+1}} \\ \underline{c}_{\xi_{t+1}} & \text{if } c_t(h^t) < \underline{c}_{\xi_{t+1}} \end{cases},$$

and there is a one-to-one correspondence between feasible U^0 and agent 1's initial consumption $c_0(h^0) \in [\underline{c}_{\xi_0}, \overline{c}_{\xi_0}]$. We can write this in the form (5) as $c_{t+1} = f(c_t, \widetilde{\xi}_t)$ where $\widetilde{\xi}_t := \xi_{t+1}$, and where

$$f(c,\xi) = \begin{cases} \overline{c}_\xi & \text{if } c > \overline{c}_\xi \\ c & \text{if } c \in I_\xi \\ \underline{c}_\xi & \text{if } c < \underline{c}_\xi \end{cases}.$$

Here, the first-best is sustainable for some U^0 if and only if $\cap_\xi I_\xi \neq \emptyset$. Kocherlakota (1996) assumes that the shocks are i.i.d. and shows (his Proposition 4.2) that if the first-best is not sustainable then the distribution of transfers converges weakly to the same non-degenerate distribution for all U^0.

Define $c_{\min} := \min_\xi \overline{c}_\xi$, $c_{\max} := \max_\xi \underline{c}_\xi$. If the first-best is not sustainable, $\cap_\xi I_\xi = \emptyset$, then $c_{\min} < c_{\max}$. If $c_t \in [c_{\min}, c_{\max}]$, $c_{t+1} = f(c_t, \widetilde{\xi}_t) \in [c_{\min}, c_{\max}]$ for all $\widetilde{\xi}_t$. Define $c := (c_{\min}+c_{\max})/2$; then if $N_1 = N_2 = 2$, $z_{1,1} \in \arg\max_\xi \underline{c}_\xi$, $z_{1,2} \in \arg\min_\xi \overline{c}_\xi$, and z_0 arbitrary, all the assumptions of Corollary 1 are satisfied (where $[c_{\min}, c_{\max}]$ replaces $[0, 1]$). Thus, there exists a unique distribution π such that the distributions of c_t converge to π in the uniform metric for any initial value $c_0 \in [c_{\min}, c_{\max}]$. Clearly, $c_t \in [\underline{c}_{\xi_0}, \overline{c}_{\xi_0}] \cup [c_{\min}, c_{\max}]$ all t, but $[\underline{c}_{\xi_0}, \overline{c}_{\xi_0}] \backslash [c_{\min}, c_{\max}]$ is transient.

If the first-best is sustainable, then the mixing condition is not satisfied. In this case it can be seen immediately that there is monotone convergence to a first-best allocation (the limit allocation is dependent on the initial condition).

4. CONCLUSION

In this paper we have established convergence results that can be used in a range of models whose dynamics can be represented by a stochastic recursion, and which satisfy two main conditions; first, for a given value of the exogenous driving process, the future value of the endogenous variable is monotone increasing in its current value; secondly, the driving process is regenerative. The latter includes as a special case irreducible finite Markov chains. These two conditions, along with a standard mixing condition, guarantee weak convergence to a unique limiting distribution.

Existing results typically assume that the driving process is i.i.d. Since the interpretation of the driving process in economic models is usually the exogenous economic environment, the i.i.d. assumption is very restrictive. When the driving process is itself a monotone Markov chain, it is often possible to establish convergence, but with further results needing to be established on the monotonicity of the endogenous variable with respect to the driving process. Our results, on the other hand, can be applied "off-the-shelf" and avoid the need for extra results in the monotone driving process case, and, in particular, do not require the process to be monotone.

The idea behind our result is that a regenerative process can be thought of as a sequence of independent and identically distributed cycles, so that we can treat each cycle itself as an i.i.d. shock and adapt earlier results from the i.i.d. case. This implies convergence looking at the sequence of regeneration points. The argument is then extended to dates other than the points of regeneration. We have illustrated our results by extending known convergence results in two canonical models to the case of Markov shocks.

APPENDIX

Proof of Theorem 1.

PROOF: The metric space of probability distributions on $[0,1]$ with metric d is complete. Due to monotonicity, it is sufficient to show that

$$d(F_n^{(0)}, F_n^{(1)}) \to 0$$

exponentially fast. Then (10) will follow.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Denote by A and B the events

$$A = \{X_N^{(1)} \leq c\} \quad \text{and} \quad B = \{X_N^{(0)} \geq c\}.$$

Note that both events are defined by $(\xi_0, \ldots, \xi_{N-1})$, i.e. belong to the sigma-algebra generated by these random variables.

The proof is by induction. For any $t \geq c$ and for any two probability measures μ and ν on $[0,1]$ with $\mu(x) \equiv \mu[0,x] \geq \nu(x) \equiv \nu[0,x]$, for all x, we may couple initial values of 4 trajectories of the Markov chain $\{X_n^{(1)}\}, \{X_n^{(\nu)}\}, \{X_n^{(\mu)}\}$, $\{X_n^{(0}\}$ in such a way that

$$1 = X_0^{(1)} \geq X_0^{(\nu)} \geq X_0^{(\mu)} \geq X_0^{(0} = 0 \quad \text{a.s.}$$

Then

$$X_n^{(1)} \geq X_n^{(\nu)} \geq X_n^{(\mu)} \geq X_n^{(0)} \quad \text{a.s. for any} \quad n,$$

and we have

$$\begin{aligned}
0 \leq F_N^{(\mu)}(t) - F_N^{(\nu)}(t) &= \mathbf{P}(X_N^{(\mu)} \leq t, A) + \mathbf{P}(X_N^{(\mu)} \leq t, \overline{A}) \\
&- \mathbf{P}(X_N^{(\nu)} \leq t, A) - \mathbf{P}(X_N^{(\nu)} \leq t, \overline{A}) \\
&= \mathbf{P}(A) + \mathbf{P}(X_N^{(\mu)} \leq t, \overline{A}) \\
&- \mathbf{P}(A) - \mathbf{P}(X_N^{(\nu)} \leq t, \overline{A}) \\
&= \int_{\overline{A}} (\mu(S^{(N)}(\overline{v}, t)) - \nu(S^{(N)}(\overline{v}, t)))\mathbf{P}((\xi_0, \ldots \xi_{N-1}) \in d\,\overline{v}) \\
&\leq \sup_x(\mu(x) - \nu(x)) \cdot \mathbf{P}(\overline{A}) \\
&\leq (1-\varepsilon) \sup_x(\mu(x) - \nu(x)).
\end{aligned}$$

Similarly, for $t < c$, we may use event B to conclude again that

$$0 \leq F_N^{(\mu)}(t) - F_N^{(\nu)}(t) \leq (1-\varepsilon) \sup_x (\mu(x) - \nu(x)).$$

Therefore,

$$\sup_t (F_N^{(\mu)}(t) - F_N^{(\nu)}(t)) \leq (1-\varepsilon) \sup_x (\mu(x) - \nu(x)).$$

Now, by induction, we obtain

$$0 \leq F_{kN}^{(0)}(t) - F_{kN}^{(1)}(t) \leq (1-\varepsilon)^k$$

for all t.

Indeed, for $k = 1$ the inequality follows from the above. Assume that it holds for $k \leq K - 1$. Then

$$\begin{aligned}
0 &\leq F_{KN}^{(0)}(t) - F_{KN}^{(1)}(t) = \mathbf{P}\left(X_{KN}^{(0)} \leq t\right) - \mathbf{P}\left(X_{KN}^{(1)} \leq t\right) \\
&= \mathbf{P}\left(X_N^{(X_{(K-1)N}^{(0)})} \leq t\right) - \mathbf{P}\left(X_N^{(X_{(K-1)N}^{(1)})} \leq t\right) \\
&\leq (1-\varepsilon) \sup_t \left(F_{(K-1)N}^{(0)}(t) - F_{(K-1)N}^{(1)}(t)\right) \leq (1-\varepsilon)^K,
\end{aligned}$$

which finishes the proof of the induction argument, and the result now follows. *Q.E.D.*

Proof of Lemma 1.

PROOF: (i) For $a > \underline{a}$, we cannot have $f(a,e) \geq a$ all $e \in E$. That is, assets will be reduced in at least one state. Suppose otherwise. Then for some $a > \underline{a}$ and for all e, where we write $v_a(a,e)$ for $\partial v(a,e)/\partial a$:

$$\begin{aligned}
v_a(a,e) &= u'(c(a,e)) \\
&= \beta R \mathrm{E}_{e'|e}\left[v_a(f(a,e),e')\right] \\
&\leq \beta R \mathrm{E}_{e'|e}\left[v_a(a,e')\right] \\
&< \mathrm{E}_{e'|e}\left[v_a(a,e')\right], \qquad \text{(A.1)}
\end{aligned}$$

where the first-line is the envelope condition, the second line follows from the first-order conditions to (15) (equality because $f(a,e) > \underline{a}$), the third line from the assumption that $f(a,e) \geq a$, which from the concavity of $v(\cdot, e)$ in a implies $v_a(f(a,e), e') \leq v_a(a, e')$ for all e', and the final line since $\beta R < 1$ and $v_a(a, e') > 0$. This is a contradiction as (A.1) cannot hold for all e (consider $e \in \arg\max_{e'} v_a(a, e')$).

(ii) Suppose that, at some (a,e) with $a > \underline{a}$, $f(a,e) \geq a$. If the same shock recurs at the next date we have $c(f(a,e), e) \geq c(a,e)$ by c increasing in its first argument, which in turn follows from $v_a(a,e) = u'(c(a,e))$ and v (respectively u) strictly concave in a (c). So consumption is at least as high, and we shall show that assets also do not decline at this date; we show that in order to satisfy the Euler equation there must be some other state with consumption sufficiently low that assets will increase after this state too, and moreover marginal utility will also have risen; by repeating this construction a contradiction will arise. As $f(a,e) > \underline{a}$, the Euler equation holds with equality (repeating lines 1 and 2 of (A.1)):

$$\text{(A.2)} \qquad u'(c(a,e)) = \beta R \mathrm{E}_{e'|e} u'(c(f(a,e), e')).$$

From $c(f(a,e), e) \geq c(a,e)$, $\beta R < 1$ and u being strictly concave, (A.2) implies that $u'(c(a,e)) \leq \beta R u'(c(f(a,e), e'))$ for some $e' \neq e$. Thus

$$c(f(a,e), e')^{-\gamma} \geq (\beta R)^{-1} c(a,e)^{-\gamma}$$

so

$$\text{(A.3)} \qquad c(f(a,e), e') \leq (\beta R)^{1/\gamma} c(a,e).$$

For

$$\text{(A.4)} \qquad c(a,e) > (e_n - e_1)/(1 - (\beta R)^{1/\gamma}),$$

we have from (A.5):

$$\begin{aligned} c(a,e) - c(f(a,e),e') &\geq (1-(\beta R)^{1/\gamma})c(a,e) \\ &> (\widehat{e}_n - \widehat{e}_1). \end{aligned} \tag{A.5}$$

Moreover, $f(a,e) \geq a$ implies $f(f(a,e),e) \geq f(a,e)$, by $f(\cdot,e)$ nondecreasing in a, that is

$$f(f(a,e),e) = R(f(a,e) + e - c(f(a,e),e)) \geq f(a,e). \tag{A.6}$$

Then if (A.4) holds,

$$\begin{aligned} f(f(a,e),e') &= R(f(a,e) + e' - c(f(a,e),e')) \\ &> R(f(a,e) + e' + (\widehat{e}_n - \widehat{e}_1) - c(a,e)) \\ &\geq R(f(a,e) + e' + (\widehat{e}_n - \widehat{e}_1) - c(f(a,e),e)) \\ &\geq f(a,e), \end{aligned} \tag{A.7}$$

where the first line follows from the budget constraint, the second from (A.5), the third from $c(f(a,e),e) \geq c(a,e)$, and the last from (A.6) given that $e - e' \leq (\widehat{e}_n - \widehat{e}_1)$.

From v bounded in a and the envelope condition $v_a(a,e) = u'(c(a,e))$, we have $\lim_{a\to\infty} c(a,e) \to \infty$, so there exists some $\widehat{a}$ such that (A.4) holds for $a > \widehat{a}$, all $e \in E$. We conclude that if $a_{(0)} > \widehat{a}$ and $f(a_{(0)}, e_{(0)}) \geq a_{(0)}$ for some $e_{(0)}$, we have $f(f(a_{(0)}, e_{(0)}), e_{(1)}) > f(a_{(0)}, e_{(0)})$ for some $e_{(1)} \neq e_{(0)}$ from (A.7). Since $f(a_{(0)}, e_{(0)}) > \widehat{a}$, we can repeat the same argument in state $e_{(1)}$ with assets $a_{(1)} = f(a_{(0)}, e_{(0)})$, and so on. This implies a sequence $(a_{(0)}, e_{(0)}), (a_{(1)}, e_{(1)}), (a_{(2)}, e_{(2)}), \ldots$, with $a_{(m+1)} > a_{(m)}$ for $m \geq 1$, and with $u'(c(a_{(m+1)}, e_{(m+1)})) \geq (\beta R)^{-1} u'(c(a_{(m)}, e_{(m)}))$ for $m \geq 0$. Clearly $e_{(m)} = e_{(p)}$ for some $1 \leq m < p \leq n+1$. Then

$$u'\left(c(a_{(p)}, e_{(m)})\right) \geq (\beta R)^{-1} u'\left(c(a_{(m)}, e_{(m)})\right),$$

so that $c(a_{(p)}, e_{(m)}) < c(a_{(m)}, e_{(m)})$, which contradicts $c(a,e)$ increasing in its first argument. *Q.E.D.*

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