STOCHASTIC STABILITY OF MONOTONE ECONOMIES IN REGENERATIVE ENVIRONMENTS

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June 2017

Abstract
We introduce and analyze a new class of monotone stochastic recursions in a regenerative environment which is essentially broader than that of Markov chains. We prove stability theorems and apply our results to three canonical models in recursive economics, generalizing some known stability results to the cases when driving sequences are not independent and identically distributed.

Keywords: Monotone Economy, Markov Chain, Stochastic Recursion, Driving Sequence, Renegerative Sequence, Existence and Uniqueness of a Stationary Distribution, Stochastic Stability, Bewley-Huggett-Aiyagari Model, Risk-Sharing Model.
JEL Classification: C61, C62.

1. introduction

This paper develops results on stochastic stability, in particular, uniform convergence to a unique stationary distribution, for a class of monotone stochastic recursions where the exogenous stochastic driving process is regenerative. A regenerative stochastic process is, loosely, a process that has independent and identically distributed (i.i.d.) cycles. We apply our results to three important workhorse models in macroeconomics. The Bewley-Imrohoroglu-Huggett-Aiyagari precautionary savings model of Bewley (1987), İmrohoroğlu (1992), Huggett (1993) and Aiyagari (1994), the one-sector stochastic optimal growth model of Brock and Mirman (1972), and the risk-sharing under limited commitment model of Kocherlakota (1996). In each of these examples, we are able to demonstrate uniqueness and stability results under less restrictive assumptions than in existing literature.

To illustrate the applicability of our approach, consider a typical problem in economic dynamics that can be solved recursively using a Bellman equation of the form

$$V(x,z) = \sup_{x' \in \Gamma(x,z)} u(x,z,x') + \beta \int V(x',z')Q(z,dz'). \tag{1.1}$$

In this equation x is an endogenous state variable, z is an exogenous shock, u is the per-period payoff function, Γ is the constraint set, Q is the transition

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function for the shock and V is the value function. Variables indicated by a $'$ are the next period values. Stochastic dynamic programming problems of this type are discussed extensively in Stokey et al. (1989).[1] When there is a unique solution to the Bellman equation, it can be described by a policy function $x' = f(x, z)$.

The policy function from dynamic problems of the type described in (1.1) are examples of a *stochastic recursive sequence (SRS)*, or *stochastic recursion* of the form

$$X_{t+1} = f(X_t, Z_t) \quad \text{a.s.,} \tag{1.2}$$

where $\{Z_t\}$ is a stochastic process with $Z_t \in \mathcal{Z}$, $X \in \mathcal{X}$ is the state variable of economic interest and $f \colon \mathcal{X} \times \mathcal{Z} \to \mathcal{X}$ is an appropriately measurable function. The process $\{Z_t\}$ is known as the *driving sequence* of the stochastic recursion. For a given X_0 and given (random) values of $Z_0, \ldots, Z_{t-1}$, the system (1.2) generates a (random) value of X_t.

It is well-known that a stochastic recursive sequence is more general than a Markov chain (see, e.g., Borovkov and Foss, 1992).[2] In particular, under extremely general conditions on the state space $\mathcal{X}$ (see Section 2.1 for details), any time-homogeneous Markov chain (equivalently, discrete-time Markov process, DTMP) may be represented as an SRS (1.2) with independent and identically distributed (i.i.d.) driving elements $Z_0, Z_1, \ldots$.

Stachurski (2009) gives a number of examples of stochastic recursions in economics including threshold models and random mutations to best responses in a co-ordination game. Other examples include linear models, such as $X_{t+1} = a_t X_t + b_t$ where $Z_t = (a_t, b_t)$ is a random vector (Horst, 2001). The focus of our applications will however, be on recursions generated from dynamic programming problems of the type in equation (1.1).

The question we address in this paper is whether there exist a unique stationary distribution for X when the driving process is regenerative. The answer to this question depends on the spaces $\mathcal{X}$ and $\mathcal{Z}$, the function f and the nature of the driving sequence. In this paper we are concerned with the case where the function f is monotone *increasing* in X and where Z is a *regenerative process*. We make appropriate assumptions on $\mathcal{X}$ and $\mathcal{Z}$ that are specified below. Loosely, a stochastic process is regenerative if it can be split into independent and identically distributed (i.i.d.) cycles. That is, if there exists a subsequence of (random) dates such that the process has the same probabilistic behavior between any two consecutive dates in the subsequence. The cycle lengths (lengths of time intervals between these dates) may also be random, in general, with the only requirement that they have a finite mean value. As an example, consider a finite-state time-homogeneous Markov chain with a single closed class of communicating states.

[1] This formulation of a control problem is sometimes described as being of the Euler class (Miao, 2014).

[2] We follow the terminology of Meyn and Tweedie (2009) and use the term Markov Chain to refer to any discrete-time Markov process (DTMP) whether the state space is finite, countable or continuous.

If the chain starts in some state z_0, then the subsequence of dates corresponds to the dates at which the chain revisits state z_0. Between each of these dates the chain has the same probabilistic behavior.[3] The class of regenerative processes is large and includes not only ergodic Markov chains, but also renewal processes, Brownian motion, waiting times in general queues and so on.[4]

Before explaining our approach in more detail, we outline three traditional approaches that are used to address stability and uniqueness issues for SRS of the type described by equation (1.2). First, when $\{Z_t\}$ is i.i.d., the process for X_t is Markov and standard existence and convergence results for discrete-time Markov processes can be applied. For example, when f is monotone in the first argument, it is well-known that there is convergence to a unique invariant distribution if a mixing or splitting condition holds (see, e.g., Bhattacharya and Majumdar, 2007; Dubins and Freedman, 1966; Hopenhayn and Prescott, 1992; Stokey et al., 1989).[5]

Second, stability results are also known in a more general setting where the driving sequence $\{Z_t\}$ is stationary or even asymptotically stationary (this literature originated with Loynes (1962), see, e.g., Borovkov and Foss (1992) and references therein). By *stationarity* we mean *stationarity in the strong sense*, that is, for any finite k, the distribution of a finite-dimensional vector $(Z_t, \dots, Z_{t+k})$ does not depend on t. The most basic result is that if the state space for the X's is partially ordered and possesses a least element, say 0, and if SRS $X_{t+1} = f(X_t, Z_t)$ starts from the bottom point $X_0 = 0$, with f monotone increasing in the first argument, then the distribution of X_t is monotone increasing in t and, given that the sequence is *tight*,[6] it converges to a limit which is the *minimal* stationary solution to recursion (1.1). In general, there may be many solutions, and for the minimal solution to be unique, one has to require additional assumptions, such as, the existence of *renovating events* (for details see, e.g., Brandt, 1985; Foss, 1983). These results seem to have been relatively little used in the economics literature although in Bewley (1987) it is assumed that there is a Markov driving sequence for shocks that starts from a stationary state.

A third situation where results are known is considered by Stokey et al. (1989, Chapter 9) and Hopenhayn and Prescott (1992). If $\{Z_t\}$ is itself a Markov chain,

[3] An i.i.d. process is one that is regenerative at every date.

[4] We are not the first to consider regenerative processes in the economics literature. For example, Kamihigashi and Stachurski (2015) consider perfect simulation of a stochastic recursion of the form $X_{t+1} = f(X_t, \xi_t)\mathbb{1}\{X_t \geq x\} + \epsilon_t \mathbb{1}\{X_t < x\}$ where $\mathcal{X} = [a, b]$, $x \in (a, b)$, f is increasing in X and $\{\xi_t\}$ and $\{\epsilon_t\}$ are i.i.d. The process regenerates for values $X_t < x$. This process arises in models of industry dynamics with entry and exit (see Hopenhayn, 1992). It is a Markov process, but it is not monotone unless the distribution of $f(x, \xi)$ stochastically dominates the distribution of ϵ.

[5] Stokey et al. (1989) use the Feller property, which is a continuity requirement, together with monotonicity and a mixing condition to derive the results. Hopenhayn and Prescott (1992) develop an existence result using monotonicity alone, and combined with a mixing condition, establish that uniqueness and stability follow.

[6] Tightness in this context means that for any $\varepsilon > 0$ there exists K_ε such that $\mathbf{P}(X_t \geq K_\varepsilon) \leq \varepsilon$ for all t.

or equivalently an SRS of the form $Z_t = g(Z_{t-1}, \varepsilon_{t-1})$ with i.i.d. $\{\varepsilon_t\}$, then $Y_t = (X_t, Z_t)$ is a time-homogeneous Markov chain, equivalently, an SRS of the form $Y_{t+1} = F(Y_t, \varepsilon_t) := (f(X_t, Z_t), g(Z_t, \varepsilon_t))$. Then, provided a mixing condition is satisfied, one can use the monotone convergence approach to establish convergence of the extended Markov chain Y_t. This is the approach generally used in the economics literature. There are however, three disadvantages to this approach. First, in order to apply monotone convergence results, it is required that function g is increasing in the first argument. That is, it is required that the driving process is itself monotone (positively correlated). Whilst this may be natural in many economic contexts, it may be restrictive in others.[7] Second, to apply monotone convergence results, it is required that function f is monotone (increasing) in both arguments, not just the first argument. This can be problematic in situations where the SRS is derived as a policy function of a dynamic programming problem. In this case, establishing monotonicity in the second argument may require extra restrictions on preferences and technology. This is the case in the one sector stochastic optimal growth model with correlated shocks that is studied by Donaldson and Mehra (1983) and others; see section 3.2. Third, the fact that the state space for the extended state variable, $\mathcal{X} \times \mathcal{Z}$, is of a larger dimension, may create additional technical difficulties and establishing that the mixing condition is satisfied may become less straightforward.

In this paper we exploit the i.i.d. cycle property of regenerative processes. We use this property to construct a Markov process defined at the regeneration times driven by an i.i.d. random variable. Together with an analogue of the monotone mixing or *splitting condition* of Bhattacharya and Majumdar (1999, condition (1.2)) this can be used to establish convergence to a unique stationary distribution.

We develop our approach in a simple scenario with a compact and completely ordered state space $\mathcal{X}$ (which may be taken to be $[a, b]$, $a, b \in \mathbb{R}$, $a < b$, without loss of generality). In the case where the driving sequence is i.i.d., the splitting condition says that (we focus here on the i.i.d. case for simplicity of notation and explanations), for some $c \in [a, b]$, there is a finite time N such that for the Markov chain $X_t^{(b)}$ that starts from the maximal state $X_0^{(b)} = b$ at time zero (with any Z_0), the probability $\mathbf{P}(X_N^{(b)} \le c) > 0$ and, second, for the Markov chain $X_t^{(a)}$ that starts from the minimal state $X_0^{(a)} = a$ at time zero (with any Z_0), the probability $\mathbf{P}(X_N^{(a)} \ge c) > 0$. In Section 2.1 we reproduce a result of Bhattacharya and Majumdar (2007) for the case where f is monotone increasing, the driving sequence is i.i.d. and the splitting condition holds (Theorem 1) that shows there is exponentially fast convergence to a unique stationary distribution. Theorem 2 in Section 2.2 extends this result to allow for a regenerative driving sequence. A corollary to this theorem (Corollary 1) is provided in Sec-

[7] For example, if the states that the driving process represents have no natural ordering, there may be no reordering of states such that the process is monotone. We give further examples below.

tion 2.3 that considers the important special case where the driving sequence is itself an aperiodic Markov chain with a positive atom. For such regenerative driving sequences, our approach generalizes the standard result whilst avoiding the disadvantages mentioned above. In particular, we establish convergence to a unique stationary distribution without needing to assume the driving process is itself monotone or that the function f is increasing in the second argument. In addition, our convergence applies directly to the state space of interest, $\mathcal{X}$, and can be extended to the joint distribution on the state space $\mathcal{X} \times \mathcal{Z}$.

The paper is organized as follows. In Section 2, we describe the model and provide our main results. First, we describe regenerative processes. Next, we review the results of Bhattacharya and Majumdar (2007) for an i.i.d. driving sequence. Then, we present the main results showing that if a mixing condition similar to that given in Bhattacharya and Majumdar (2007) are satisfied between the dates when the driving sequence regenerates, then stability holds. Section 3 presents the three economic applications of our main result to an income fluctuation problem with savings (Section 3.1), stochastic optimal growth (Section 3.2) and risk sharing with limited commitment (Section 3.3). The proofs of the main and other subsidiary proofs are put in the Appendix.

2. THE MAIN MODEL

In this section, we outline the main properties of discrete-time regenerative processes, provide our lead example of regeneration for Markov chains, and introduce our main model, which is a stochastic recursive sequence with a regenerative driver.

Let $Z_t, t = 0, 1, \ldots$ be a (one-sided) regenerative sequence on a general measurable space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$. The sequence is *regenerative* if there exists an increasing sequence of integer-valued random variables (times) $0 = T_{-1} \le T_0 < T_1 < T_2 < \ldots$ such that, for $\tau_n = T_n - T_{n-1}, n \ge 0$, the vectors

$$(2.1) \qquad \{\tau_n, Z_{T_{n-1}}, \ldots, Z_{T_n - 1}\}$$

are independent for $n \ge 0$ and identically distributed for $n \ge 1$. A random vector (2.1) is called a *cycle* with *cycle length* τ_n and with $\{Z_{T_{n-1}}, \ldots, Z_{T_n - 1}\}$ the sequence of "shocks" over the cycle starting at the regenerative time T_{n-1} and up to the period before the next regenerative time, i.e., $T_n - 1$.

Furthermore, we assume that

$$(2.2) \qquad \mathbf{E}\tau_1 < \infty.$$

It is known (see, e.g., Asmussen, 2003) that if, in addition, regenerative times are *aperiodic*,

$$(2.3) \qquad G.C.D.\{n : \mathbf{P}(\tau_1 = n) > 0\} = 1,$$

then Z_t has a unique stationary distribution,[8] say π, and converges to it in the total variation norm:

$$\sup_{B\in\mathcal{B}_{\mathcal{Z}}} |\mathbf{P}(Z_t \in B) - \pi(B)| \to 0, \quad \text{a.s.} \quad t \to \infty.$$

The main aim of the paper is to study the behavior of a recursive sequence

$$X_{t+1} = f\left(X_t, Z_t\right), \quad t = 0, 1, \ldots, \tag{2.4}$$

that starts from $X_0 = x \in \mathcal{X}$, assuming that

- the function f is measurable and is *monotone* in the first argument, with respect to some ordering;
- sequence $\{Z_t\}$ is regenerative and satisfies conditions (2.2)-(2.3).[9]

Example 1 The simplest possible example of a regenerative process is when $\{Z_t\}$ is an i.i.d. process. In this case $T_n = n$ and $\tau_n = 1$ for $n \geq 1$. All cycles are of length one.

Example 2 In many economic applications the driving process is modeled as a time-homogenous, irreducible and aperiodic Markov chain $\{Z_t\}$ taking values in a finite state space Z. In this case we can pick any particular state z_0 and then every time the process returns to z_0, a new sequence is formed from the states occurring until z_0 is visited again. The regeneration times $T_0 < T_1 < T_2 \ldots$ are the hitting times of z_0. By the Markov property, these sequences and their length are independent and identically distributed. Similarly, the hitting times are aperiodic and (2.2)-(2.3) are satisfied.

Example 3 Example 2 is easily generalized to a positive recurrent time-homogeneous Markov chain with a general state space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ that has a positive atom. A Markov chain has a positive atom if there is a point $z_0 \in \mathcal{Z}$ such that, for any $z \in \mathcal{Z}$,

$$T_1^z = \min\{t\ : Z_t = z_0 \mid Z_0 = z\} < \infty \quad \text{a.s.}$$

and

$$\mathbf{E}T_1^{z_0} < \infty.$$

[8]Throughout we use the term unique stationary distribution and in our context this equivalent to a unique limiting distribution for any initial value of the process. Other terms used for stationary distribution are invariant and steady-state distribution.

[9]In an earlier version of the paper (available on ArXiV, see Foss et al. (2016)) we allow for an extension where the recursive sequence is $X_{t+1} = f\left(X_t, \xi_t^{Z_t}\right)$ in which $\{\xi_t^z\}_{z\in\mathcal{Z},-\infty<t<\infty}$ are a family of mutually independent random variables and where for each $z \in \mathcal{Z}$, $\{\xi_t^z\}_{t\geq 1}$ are i.i.d. with a common distribution. f is again assumed monotone in its first argument. It is shown that our main theorem holds with this more general driving process.

Again the regeneration times $T_0 < T_1 < T_2 \ldots$ are the hitting times of z_0. By the Markov property, these sequences and their length are independent and identically distributed. Provided these hitting times are additionally assumed to be aperiodic, then (2.2)-(2.3) are satisfied.

Most of the known results on stability for stochastic recursions are for the case where the driving process is i.i.d. or the driving process is Markov *and* monotone increasing. Our extension is to provide similar stability results for any regenerative process including Markov processes that are not monotonic. The applications we consider in Sections 3.2 and 3.3 are with Markov driving processes as in Example 2 and the application considered in Section 3.1 is with a driving process defined on a general state space as in Example 3. Similarly, models where a potentially non-Markov process drives an agent's environment, but it periodically returns to some initial state, can be incorporated into our framework, as in the next example.

Example 4 A worker who has just entered the unemployment pool at $t = 0$ receives unemployment benefit b until successfully matched with a firm, thereafter receiving wages w_t until a separation occurs, whereupon the worker returns to the initial unemployment state (i.e., as at date 0). Wages and the matching and separation hazards evolve jointly according to a general stochastic process. Formally let $\{E_t, y_t\}$ represent the process where $E_t \in \{0, 1\}$ represents employment status (0 for unemployed, 1 for employed) and y_t is income at time t ($y_t = b$ when $E_t = 0$), and $E_0 = 0$. Then, $\{E_t, y_t\}$ is a regenerative process with regenerative times $\{T_j\}$ given by each time the worker transitions from employment to unemployment: $T_{-1} = T_0 = 0$, $T_1 = \min\{t > 0 : E_{t-1} = 0, E_t = 1\}$, the first time the worker returns to unemployment, and likewise for each $j = 2, \ldots$, let

$$T_j = \min\{t > T_{j-1} : E_{t-1} = 0, E_t = 1\}.$$

Then, provided the mean return time to the initial state is finite and the return times are mutually independent and have an aperiodic distribution (e.g., if transition probabilities are positive at each date), assumptions (2.2)-(2.3) are satisfied.

In the rest of this section, we first consider the standard case with an i.i.d. driving process. In Section 2.2 we provide the result of our main theorem for a regenerative driving process. In Section 2.3 we specialize our result to the case where the driving process is a Markov chain with a countable state space and a positive recurrent atom. Finally, in Section 2.4 we discuss our results in relation to some of the existing literature on monotone economies.

2.1. *I.i.d. driving sequence*

We start with a particular case when Z_t is i.i.d. We revisit some results from Bhattacharya and Majumdar (2007) (see also Dubins and Freedman (1966)).

The relation between time-homogeneous Markov chains (with a general measurable state space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$) and recursions (2.4) with i.i.d. drivers is well-understood (see, e.g., Borovkov and Foss (1992); Kifer (1986)): if the sigma-algebra $\mathcal{B}_{\mathcal{X}}$ is countably generated, then a Markov chain may be represented as a stochastic recursion (2.4) with an i.i.d. driving sequence $\{Z_t\}$. In particular, any real-valued or vector-valued time-homogeneous Markov chain may be represented as a stochastic recursion (2.4).

In what follows, we restrict our attention to real-valued X_t and, moreover, assume that

$$\text{the state space} \quad \mathcal{X} \quad \text{is the closed interval} \quad [a, b]. \tag{2.5}$$

We define the *uniform, or Kolmogorov distance* between probability distributions on the real line as

$$d(F, G) = \sup_x |F(x) - G(x)|. \tag{2.6}$$

Here $F(x) = F(-\infty, x]$ and $G(x) = G(-\infty, x]$ are the distribution functions. Let $F(x-) = F(-\infty, x)$ and $G(x-) = G(-\infty, x)$.[10] Then, by the right-continuity of distribution functions,

$$d(F, G) = \sup_x |F(x-) - G(x-)| \equiv \sup_x \max\left(|F(x-) - G(x-)|, |F(x) - G(x)|\right). \tag{2.7}$$

Next, we assume the function f to be *monotone increasing* in the first argument: for each $z \in \mathcal{Z}$ and for each $a \le x_1 \le x_2 \le b$,

$$f(x_1, z) \le f(x_2, z).$$

We write for short

$$\mathbf{P}^{(x)}(\cdot) = \mathbf{P}(\cdot \mid X_0 = x).$$

We also denote by $F_t^{(x)}$ the distribution function of the random variable X_t if $X_0 = x$ (and more generally denote by $F_t^{(\mu_0)}$ the distribution function of X_t if X_0 has distribution μ_0). Our first Theorem reproduces a result of Bhattacharya and Majumdar (2007).[11] It shows convergence of the process X_t to a unique stationary distribution under a monotone mixing or splitting condition. Recall that a distribution, say π, is *stationary* for a Markov chain X_t, $t = 0, 1, \ldots$ if taking the initial value X_0 with distribution π implies that all X_t, $t \ge 1$ also have distribution π. Results of this type were originally obtained in Dubins and Freedman (1966) (under an additional assumption of continuity of the mapping f).

[10] Note that convergence in the uniform distance is weaker than convergence in the total variation norm.

[11] An improved version of the proof of this result of Bhattacharya and Majumdar (2007) is provided in an earlier version of the paper (Foss et al. (2016)).

THEOREM 1 *Assume that time-homogeneous Markov chain X_t is represented by the stochastic recursion (2.4) with i.i.d. driving sequence $\{Z_t\}$, where function $f : [a,b] \times \mathcal{Z} \to [a,b]$ is monotone increasing in the first argument.*
Assume there exists a number $c \in [a,b]$ and integer $N \geq 1$ such that

$$\varepsilon_1 := \mathbf{P}^{(b)}(X_N \leq c) > 0$$

and

$$\varepsilon_2 := \mathbf{P}^{(a)}(X_N \geq c) > 0.$$

Then, there exists a distribution π on $[a,b]$ such that, for any initial distribution μ_0,

$$\sup_x d(F_t^{(\mu_0)}, \pi) \to 0, \quad t \to \infty \tag{2.8}$$

exponentially fast.
Furthermore, π is the unique stationary distribution for the Markov chain X_t.

REMARK 1 Theorem 1 is easily generalized to a case where the set $\mathcal{S}$ has a partial order, $\leq$, such that there exists a least element $s_0 \in \mathcal{S}$ and greatest element $s_1 \in \mathcal{S}$ and f is monotone increasing in the first argument (with respect to the partial order $\leq$).[12]

2.2. *Regenerative driving process*

We now turn our attention to the general regenerative setting (2.4), but continue to assume (2.5) to hold, that is, that the state space $\mathcal{X}$ is a closed interval.[13]

We generalize Theorem 1 to this setting. The way this is done is first to apply Theorem 1 to the regeneration times using the i.i.d. nature of the cycles between the regeneration times. This implies convergence to a distribution π at the regeneration times. Next, convergence for all dates can be established using the fact that the probabilistic nature of all cycles after the first is the same and that each cycle will in the limit start from the same distribution π. This stationary distribution for X_t, say μ may, in general, differ from π and we give a simple example below (Example 5) where they do differ.

To proceed with the first step we introduce an auxiliary process $\widetilde{X}_t^{(\alpha)}$ that starts from $\widetilde{X}_0^{(\alpha)} = \alpha$ at time 0, and follows the recursion

$$\widetilde{X}_{t+1}^{(\alpha)} = f\left(\widetilde{X}_t^{(\alpha)}, Z_{T_0+t}\right) \quad \text{for all} \quad t \geq 0.$$

[12] In this case, the mixing condition requires that there exists an $\varepsilon > 0$, an integer $N \geq 1$ and sets $\mathcal{C}_u \subset \mathcal{S}$ and $\mathcal{C}_l \subset \mathcal{S}$ such that for every element $s \in \mathcal{S}$, there either exists an element $c \in \mathcal{C}_u$ such that $s \geq c$, or there exists an element $c \in \mathcal{C}_l$ such that $s \leq c$; and for every $c \in \mathcal{C}_u$, $\mathbf{P}^{(s_1)}(X_N \leq c) > \varepsilon$, and for every $c \in \mathcal{C}_l$, $\mathbf{P}^{(s_0)}(X_N \geq c) > \varepsilon$.

[13] This is less restrictive than it may seem because even when the state space is unbounded, it may be possible to show that all states outside of the closed interval are transient and the state must end up in the closed interval.

The auxiliary process $\widetilde{X}_t^{(\alpha)}$ coincides in distribution with the process X started at time T_0 (i.e., at the start of the first full cycle) from the state α, and assumptions (2.9) and (2.10) below ensure the mixing (similar to that guaranteed by conditions of Theorem 1) over a typical cycle (from T_0 to T_1) of the regenerative process Z. More generally, we consider an auxiliary process $\widetilde{X}_t^{(F)}$ that follows the recursion

$$\widetilde{X}_{t+1}^{(F)} = f\left(\widetilde{X}_t^{(F)}, Z_{T_0+t}\right) \quad \text{for all} \quad t \geq 0$$

and that starts from a random variable $\widetilde{X}_0^{(F)}$ that has distribution F (and which does not depend on random variables $\{Z_{T_0+t}, t \geq 0\}$.Denote by $f^{(k)}$ the k-th iteration of function f, so $f^{(1)} = f$ and, for, $k \geq 1$,

$$f^{(k+1)}(x, u_1, \ldots, u_{k+1}) = f\left(f^{(k)}(x, u_1, \ldots, u_k), u_{k+1}\right),$$

and let $f^{(0)}$ be the identity function.

Theorem 2 *Assume that recursive sequence* $\{X_t\}$ *is defined by* (2.4) *where the function* f *is monotone increasing in the first argument and the sequence* $\{Z_t\}$ *is regenerative with regenerative times* $\{T_n\}$ *that satisfy conditions* (2.2)-(2.3).

Assume that the following conditions hold:

$$\varepsilon_1 := \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(b)} \leq c\right) > 0, \tag{2.9}$$

and

$$\varepsilon_2 := \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(a)} \geq c\right) > 0. \tag{2.10}$$

Then there exists a distribution π *on* $[a, b]$ *such that*

$$\rho_t := \sup_x d(G_n^{(x)}, \pi) = \sup_x \sup_r |G_n^{(x)}(r) - \pi(-\infty, r]| \to 0, \quad n \to \infty \tag{2.11}$$

exponentially fast. Here $G_n^{(x)}$ *is the distribution of* X_{T_n} *if* $X_{T_0} = x$.

Furthermore, the distributions of X_t *converge in the uniform metric to distribution*

$$\mu(\cdot) = \frac{1}{\mathbf{E}(\tau_1)} \sum_{l=0}^{\infty} \mathbf{P}\left(\tau_1 > l, f^{(l)}\left(\widetilde{X}_0^{(\pi)}, Z_{T_0}, \ldots, Z_{T_0+l-1}\right) \in \cdot\right)$$

for any initial value X_0.

The following also holds for the joint distributions of (X_t, Z_t)*:*

$$\sup_r \sup_{A \in \mathcal{B}_{\mathcal{Z}}} \Bigg| \mathbf{P}(X_t \leq r, Z_t \in A) - \frac{1}{\mathbf{E}(\tau_1)} \sum_{l=0}^{\infty} \mathbf{P}\left(\tau_1 > l, f^{(l)}\left(\widetilde{X}_0^{(\pi)}, Z_{T_0}, \ldots, Z_{T_0+l-1}\right) \leq r, Z_{T_0+l-1} \in A\right)\Bigg| \to 0$$

as $t \to \infty$, *for any initial value* X_0.

REMARK 2 Note that, as in the Markovian case of Theorem 1, we do not require the function f to be continuous in the first argument.

REMARK 3 In general, we require only the first moment of τ_1 to be finite, so convergence in the regeneration theorem may be arbitrarily slow, and the same holds for convergence of the distribution F_t of random variable X_t to μ. However, if τ_1 has finite $(1+r)$-th moment, then $d(F_t, \mu)$ decays no slower than t^{-r}; and if τ_1 has finite exponential moment, then the convergence is exponentially fast.

REMARK 4 The mixing conditions (2.9)-(2.10) are required to apply over a single regenerative cycle. However this is not restrictive as a new cycle can be defined for example to consist of appropriate multiple occurrences of an original cycle.

The following simple example illustrates an application of the theorem and computation of the limiting distribution. It also shows that the distributions π and μ in Theorem 2 may, in general, be different.

EXAMPLE 5 Consider a simple example, with only two states of environment $\mathcal{V} = \{1, 2\}$ and with four-state space $\mathcal{X} = \{0, 1, 2, 3\}$ (i.e., $[a, b] = [0, 3]$). Assume sequence $\{V_t\}$ to be regenerative, with the typical cycle taking two values, $(2, 1)$ and $(2, 2, 1)$, with equal probabilities $1/2$, so the cycle length τ_1 is either 2 or 3, with mean $\mathbf{E}\tau_1 = 5/2$. Let $\{\xi_t^1\}$ and $\{\xi_t^2\}$ be two mutually independent i.i.d. sequences with the following distributions: $\mathbf{P}(\xi_t^1 = k) = 1/4$ for $k = 0, 1, 2, 3$ and $\mathbf{P}(\xi_t^2 = -1) = \mathbf{P}(\xi_t^2 = -2) = 1/2$. Now define the driving sequence Z_t as $Z_t = \xi_t^{V_t}$. The stochastic recursion is given by

$$X_{t+1} = \min(3, \max(0, X_t + Z_t)), \quad t = 0, 1, \ldots.$$

It may be easily checked that the SRS satisfies all the conditions of the previous theorem.

Introduce the embedded Markov chain $Y_n = X_{T_n}$, as in the proof of the previous theorem. It is irreducible with transition probability matrix $P = \{p_{i,j}, 0 \leq i, j \leq 3\}$ given by

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{16} & \frac{1}{4} & \frac{1}{4} & \frac{5}{16} \\ \frac{3}{32} & \frac{3}{16} & \frac{1}{4} & \frac{15}{32} \end{pmatrix}.$$

For example, here

$$\begin{aligned} p_{3,1} &= \mathbf{P}(\tau_1 = 2, \xi_1^2 = -2, \xi_2^1 = 0) + \mathbf{P}(\tau_1 = 3, \xi_1^2 = \xi_2^2 = -1, \xi_3^1 = 0) \\ &\quad + \mathbf{P}(\tau_1 = 3, \xi_1^2 + \xi_2^2 = -3, \xi_3^1 = 1) + \mathbf{P}(\tau_1 = 3, \xi_1^2 = \xi_2^2 = -2, \xi_3^1 = 1) \\ &= \frac{1}{16} + \frac{1}{32} + \frac{1}{16} + \frac{1}{32} = \frac{3}{16}. \end{aligned}$$

Then the distribution of Y_n converges to $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ which may be found by solving $\pi P = \pi$ with $\sum \pi_i = 1$. So we get $\pi = (29/160, 183/800, 1/4, 17/50)$. Furthermore, the limiting distribution for X_t is given by

$$\mu_k = \frac{1}{\mathbf{E}\tau_1}(\mathbf{P}(Y^{(0)} = k) + \mathbf{P}(\max(0, Y^{(0)} + \xi_0^2) = k) + \mathbf{P}(\max(0, Y^{(0)} + \xi_0^2 + \xi_1^2) = k, \tau_1 = 3)),$$

for $k = 0, 1, 2, 3$, where $Y^{(0)} \sim \pi$. In particular, $\mu_3 = 2\pi_3/5$, $\mu_2 = \frac{2}{5}(\pi_2 + \pi_3/2)$, and $\mu_1 = \frac{2}{5}(\pi_1 + (\pi_2 + \pi_3)/2 + \pi_3/8) = \frac{2}{5}(\pi_1 + \pi_2/2 + 5\pi_3/8)$.

2.3. *The case where the governing sequence is Markov*

In the particular case where $\{Z_t\}$ is a Markov chain on a countable state space, Theorem 2 leads to the following corollary, which is important for two of the examples considered in the next section.

Corollary 1 *Assume again that the recursive sequence $\{X_t\}$ is defined by (2.4), and that the function f is monotone increasing in the first argument. Assume in addition that $\{Z_t\}$ is an aperiodic Markov chain on a countable state space with a positive recurrent atom at point z_0. Assume also that there exists a number $a \le c \le b$, positive integers N_1 and N_2 and sequences $z_{1,1}, \ldots, z_{N_1,1}$ and $z_{1,2}, \ldots, z_{N_2,2}$ such that $z_{N_1,1} = z_{N_2,2} = z_0$ and, for $i = 1, 2$, the following hold:*

$$p_i := \mathbf{P}(Z_j = z_{j,i}, \ \textit{for} \ \ j = 1, \ldots, N_i \mid Z_0 = z_0) > 0$$

and that

$$\delta_1 := \mathbf{P}(\widetilde{X}_{N_1}^{(b)} \le c \mid Z_0 = z_0, \ Z_j = z_{j,1}, \ j = 1, \ldots, N_1) > 0$$

and

$$\delta_2 := \mathbf{P}(\widetilde{X}_{N_2}^{(a)} \ge c \mid Z_0 = z_0, \ Z_j = z_{j,2}, \ j = 1, \ldots, N_2) > 0.$$

Then the distribution of X_t converges in the uniform metric to a unique stationary distribution.
In addition, There exists a stationary sequence (X^t, Z^t) such that, as $t \to \infty$,

$$\sup_{a \le x \le b} \sup_{B \in \mathcal{B}_{\mathcal{Z}}} |\mathbf{P}(X_t \le x, Z_t \in B) - \mathbf{P}(X^t \le x, Z^t \in B)| \to 0.$$

Remark 5 For simplicity, we have assumed that the Markov chain in Corollary 1 is defined on a countable state space with a positive recurrent atom. However, Corollary 1 can be extended to the case of a driving Markov chain on a general state space provided a "Harris-type" condition is satisfied. Here we outline the conditions required. Consider again a recursive sequence $\{X_t\}$ with the function f monotone increasing in the first argument. Assume that there exists a measurable set A in the state space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ that is *positive recurrent*:

$$T_1(z_0) = \min\{t > 0 \; : \; Z_t^{(z_0)} \in A\} < \infty \quad \text{a.s., for any} \quad z_0 \in \mathcal{Z}$$

and

$$\sup_{z_0 \in A} \mathbf{E} T_1(z_0) < \infty.$$

Here $Z_t^{(z_0)}$ is a Markov chain with initial value $Z_0^{(z_0)} = z_0$. Furthermore, assume that there exist positive integers N_1 and N_2, a positive number $p \leq 1$ and a probability measure φ on A such that, for $i = 1, 2$ and all $z_0 \in A$,

$$\mathbf{P}(Z_{N_i}^{(z_0)} \in \cdot) \geq p\varphi(\cdot)$$

and that there exists a number $a \leq c \leq b$ and positive numbers δ_1 and δ_2 such that

$$\mathbf{P}(\widetilde{X}_{N_1}^{(b)} \leq c \mid Z_0 = z_0, Z_{N_1} = z_1) \geq \delta_1$$

and

$$\mathbf{P}(\widetilde{X}_{N_2}^{(a)} \geq c \mid Z_0 = z_0, Z_{N_2} = z_2) \geq \delta_2,$$

for φ-almost surely all $z_0, z_1, z_2 \in A$. With all these conditions and aperiodicity of the Markov chain, it can be shown that the distribution of X_t converges in the uniform metric to the unique stationary distribution.

2.4. *Discussion*

In this section we consider the relation of our results to other approaches in the literature. In particular, Kamihigashi and Stachurski (2014) and Szeidl (2013), develop convergence results for Markov chains, where the driving process is i.i.d., for more general state spaces.

First note that the monotone mixing condition of Theorem 1 may be rewritten in an equivalent form as follows: assuming in addition that the trajectories $X_t^{(b)}$ and $X_t^{(a)}$ are mutually independent, there is a positive N such that $\mathbf{P}(X_N^{(b)} \leq X_N^{(a)}) \geq \delta > 0$. This condition is called a *strong reversing condition* by Kamihigashi and Stachurski (2014) because then, due to the monotonicity, it also holds for any other pair of initial conditions $a \leq x_0 < y_0 \leq b$, with the same δ and N, namely $\mathbf{P}(X_N^{(y_0)} \leq X_N^{(x_0)}) \geq \delta > 0$.

Now suppose that the state space may not contain top and/or bottom points (then the δ and N may, in general, depend on (x_0, y_0)) and, moreover, that the order is only partial. In particular, Szeidl (2013) suggested a reasonable "replacement", say, for a top point (if one does not exist) by a random "top" point. Namely, assume, say, the state space for the Markov chain is the positive half-line $[0, \infty)$ where there is the minimal element 0 but there is no maximal element, and suppose that a Markov chain X_t is defined by a stochastic recursion $X_{t+1} = f(X_t, \xi_t)$ with i.i.d. $\{\xi_t\}$. Assume that there exists a random measure μ on $[0, \infty)$ such that if $X_0 \sim \mu$ and if X_0 does not depend on ξ_0, then $X_1 = f(X_0, \xi_0)$ is *stochastically smaller* than X_0 (this means $\mathbf{P}(X_1 \le x) \ge \mathbf{P}(X_0 \le x)$, for all x). Then the distribution μ may play a role of a new random "top" point if, for example, the distribution of μ has an unbounded support. For instance, if there exists another function, say h such that $f(x, y) \le h(x, y)$ for all x, y and that a Markov chain $Y_{t+1} = h(Y_t, \xi_t)$ admits a unique stationary distribution, say μ. If μ can be easily found/determined, it may play the role of a random "top" point.

Here is a simple example. Assume that X_t is a discrete-time birth-and-death-process with immigration at 0, i.e. a non-negative integer-valued Markov chain, which is homogeneous in time and with transition probabilities $\mathbf{P}(X_1 = 1 \mid X_0 = 0) = 1 - \mathbf{P}(X_1 = 0 \mid X_0 = 0) = p_0 > 0$ and, for $k = 1, 2, \ldots$, let $\mathbf{P}(X_1 = k + 1 \mid X_0 = k) = 1 - \mathbf{P}(X_1 = k - 1 \mid X_0 = k) = p_k$. Furthermore assume that the p_k are non-decreasing in k (this makes the Markov chain monotone), that all are smaller than 1/2 and, moreover, that $\lim_{k \to \infty} p_k = p < 1/2$. Consider a Markov chain Y_t with transition probabilities $\mathbf{P}(Y_1 = k + 1 \mid Y_0 = k)) = p = 1 - \mathbf{P}(Y_1 = \max(0, k - 1)) \mid Y_0 = k)$. Then this Markov chain has a unique stationary distribution μ (which is clearly geometric), and it gives a random "top" point. In Borovkov and Foss (1992), a similar concept of a stationary majorant was developed and studied, where the top sequence $\{X_t\}$ is assumed to be stationary.

Similar ideas have been developed earlier in the context of "perfect simulation", with the introduction of an artificial random "top" point (see, e.g., (Foss and Tweedie, 1998) and (Corcoran and Tweedie, 2001) and the references therein) for simulation "from the past".

Finally, Acemoğlu and Jensen (2015) consider comparative static properties in similar setting with large numbers of agents (similar to the application we consider in Section 3.1) and a Markov driving process. Their focus however, is on developing results for any equilibrium distribution and not in establishing uniqueness.

3. Applications

In this section we present three workhorse models. In the first we allow for general driving process as in Example 3. In the second and third we assume that the driving process is a Markov chain and apply Corollary 1. We are thus able to extend known stability results in these models.

3.1. *Bewley-Imrohoroglu-Huggett-Aiyagari Precautionary Savings Model*

The basic income fluctuation model in which many risk-averse agents self-insure against idiosyncratic income shocks through borrowing and saving using a risk-free asset, is designated by Heathcote et al. (2009) "the standard incomplete markets model" and is the workhorse model in quantitative macroeconomics. At its heart is the stochastic savings models of Huggett (1993) with an exogenous borrowing constraint, or close variants of this model.[14] As Heathcote et al. (2009) observe, there are "few general results that apply to this class of problems." Existing published work in the standard model requires either that income fluctuations are i.i.d., or that an individual's income process satisfies "persistence": a higher income today implies that income tomorrow is higher in the stochastic dominance sense. That is, that the income process is monotone. Huggett (1993) has a two-state process for income and uses the Hopenhayn and Prescott (1992) approach to prove convergence of the asset distribution to a unique invariant distribution.[15] In the case of two income states, the assumption of persistence in the income process is probably innocuous. However, it may be restrictive in other cases. Obvious examples of non-monotone processes would include termination pay where a worker receives a large one-off redundancy payment followed by a long spell of unemployment, or health shocks where an insurance payout is received but future employment prospects are diminished.[16] In what follows we consider Huggett's model with a potentially uncountable number of states and dispense with the assumption that the income process is monotone (we maintain all his other assumptions). Applying our methodology, we show convergence in the uniform metric to a unique invariant distribution.[17,18]

[14] Bewley (1987) and Aiyagari (1994) vary the context but the individual savings problem is similar. They each derive existence and convergence results under slightly different assumptions. Bewley (1987) assumes that the endowment shocks are stationary Markov, Huggett (1993) assumes positive serial correlation and two states, and Aiyagari (1994) assumes that endowment shocks are i.i.d. İmrohoroğlu (1992) uses numerical methods with a two state persistent income process as in Huggett (1993).

[15] Miao (2002) extends Huggett's model from two states to many states.

[16] As another example, consider the case where there are a group of entrepreneurs who have very high income. It may be possible that these entrepreneurs have a higher chance to fall to very low income levels than those on medium income levels. This is the situation described by Kaymak and Poschke (2016) who use information from observed distributions of income and wealth to construct a transition matrix for income. The transition matrix they use does not satisfy monotonicity.

[17] In the subsequent analysis, we follow Huggett and assume that the gross interest rate, R, is fixed. This is an ingredient into finding the equilibrium rate at which assets are in zero net demand.

[18] In independent work, and in a more general context, Açikgöz (2016, Proposition 5) shows using different methods that if the income process is a finite (irreducible aperiodic) Markov chain, and there exists a "worst" positive probability sequence of incomes which is dominated at each date by any other positive probability sequence (e.g., if the lowest income state recurs with positive probability), then there exists a unique stationary distribution. Zhu (2017), who considers an income fluctuation model with endogenous labor supply, uses a related argument under the assumption that the finite Markov chain has strictly positive transition probabilities;

Agents maximize expected discounted utility

$$\mathbf{E}\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right],$$

where $c_t \in \mathbb{R}_+$ is consumption at time t, $t = 0, 1, \ldots$, $u(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma > 1$, subject to a budget constraint at each date

$$c + R^{-1}x^+ \leq x + e, \tag{3.1}$$

a borrowing constraint $x^+ \geq \underline{x}$, where e is the current endowment, x is current assets, x^+ is assets next period, c is consumption, $\beta \in (0,1)$ is the discount factor and $R^{-1} > \beta$ is the price of next-period assets. The individual's endowment at time t, e_t, is drawn from a set $E = [\underline{e}, \bar{e}]$, where $\infty > \bar{e} > \underline{e} > 0$; e_t is governed by an aperiodic *positive recurrent Markov chain with an atom,* with aperiodic regenerative times, as in Example 3, where we denote by $Q : E\times\, \mathcal{E} \to [1,0]$ the (stationary) transition function, with $\mathcal{E}$ the Borel sets of E, and we assume that the Feller property is satisfied (see, e.g., Stokey et al., 1989, ch.8). The borrowing constraint satisfies $\underline{x} < 0$ and $\underline{x} + \underline{e} - \underline{x}R^{-1} > 0$. The initial values $e_0 \in E$ and $x_0 \geq \underline{x}$ are given.

The individual's decision problem can be represented by the functional equation:

$$v_t(x, e_t) = \max_{(c,x^+)\in\Gamma(x,e_t)} u(c) + \beta\mathbf{E}_{h_{t+1}}\left[v_{t+1}(x^+, e_{t+1}) \mid e_t\right] \tag{3.2}$$

where $\mathbf{E}_{e^+}$ is expectation over e^+, $v(x, e)$ are the value functions, and

$$\Gamma(x, e) = \left\{(c, x^+) \mid c + R^{-1}x^+ \leq x + e, x^+ \geq \underline{x}, c \geq 0\right\}$$

is the constraint set. The resulting policy functions are denoted $c = c(x, e)$ and $x^+ = f(x, e)$ (i.e., an optimal policy must satisfy these a.s.). Huggett (1993, Theorem 1) proves that there is a unique, bounded and continuous solution to (3.2) and each $v(x, e)$ is increasing, strictly concave and continuously differentiable in x, while f is continuous and nondecreasing in x, and (strictly) increasing whenever $f(x, e) > \underline{x}$. These results extend to our context with a larger state space; see Miao (2002).

Huggett assumes monotonicity of the endowment process: with two endowment states, $E = \{\underline{e}, \bar{e}\}$, this means $p(\underline{e}, \underline{e}) \geq p(\bar{e}, \underline{e})$ where $p(e, e')$ denotes the transition probability. He shows that for a given R, there exists a unique stationary probability measure for $x = (x, e)$ and that there is weak convergence to this distribution for any initial distribution on x (see Huggett (1993, Theorem 2)).

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results also hold for the case where $\beta R = 1$.

We can extend this result to our more general context (non-discrete state space, no monotonicity assumption) using the following (the proof can be found in the Appendix).[19]

LEMMA 1 *There exists* $\widehat{x} \geq \underline{x}$ *such that for all* $x > \widehat{x}$*, all* $e \in E$*,* $f(x,e) < x$*.*

Given this, we can restrict attention to $[\underline{x}, \hat{x}]$ and convergence follows from the following argument.[20] Starting from $\hat{x}$, there must be some positive probability of hitting the credit constraint: given $R^{-1} > \beta$ the only reason for holding assets above $\underline{x}$ is the precautionary one, and never hitting $\underline{x}$ would imply that assets are excessive, so $\underline{x}$ must be hit at some time T with positive probability. Because $f(x,e)$ is nondecreasing in x, starting at $\underline{x}$ instead of at $\hat{x}$ but with the same sequence of endowment shocks, implies that assets at T are also at $\underline{x}$. This implies that the mixing condition of Theorem 2 is satisfied at the end of a regenerative cycle suitably defined (by the next occurrence of the atom after T). Thus there exists a unique distribution π on $[\underline{x}, \hat{x}]$ such that the distributions of x_t converge to π in the uniform metric for any initial value $x_0 \in [\underline{x}, \hat{x}]$.

3.2. *One-Sector Stochastic Optimal Growth Model*

The Brock-Mirman (Brock and Mirman, 1972) one-sector stochastic optimal growth model has been extended to the case of correlated production shocks by Donaldson and Mehra (1983) and Hopenhayn and Prescott (1992, pp. 1402–03). With correlated productivity shocks, it is possible to prove uniqueness and convergence results using the methods of Hopenhayn and Prescott (1992) or Stokey et al. (1989, Chapter 12) provided the policy function for the investment is itself monotonic in the productivity shock. Although the assumption of correlated shocks is very reasonable in this context, establishing that the policy function is monotone in the productivity shock is, as pointed out by Hopenhayn and Prescott (1992, pp. 1403), difficult without imposing very restrictive assumptions. The reason is simple. A good productivity shock today increases current output, which may allow increased investment. However, because shocks are positively correlated, output will also be higher on average tomorrow and hence consumption can be too. Therefore, it may be desirable to increase current consumption by more than the increase in current output, cutting back on current investment.[21] Since our results do not require monotonicity of the policy function in the driving process, we can establish convergence to a unique invariant distribution without requiring any extra restrictive conditions on preferences

[19] A similar result is established in Huggett (1993), and in Miao (2002) for the many state case, but using monotonicity.

[20] The details of the argument are presented in the Appendix.

[21] The sufficient condition given in Hopenhayn and Prescott (1992) for monotonicity of the policy function in the productivity shock is

$$\frac{f_{kz}}{f_k \cdot f_z} \geq -\frac{u''}{u'},$$

and productivity beyond those normally assumed in the stochastic growth model. In addition, of course, we do not require the productivity shocks to be positively correlated.

We consider a version of the Brock-Mirman one sector stochastic optimal growth model with full depreciation of capital. Paths for consumption, c_t, and capital, k_t, are chosen to

$$\max \mathbf{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$f(k_t, z_t) \geq c_t + k_{t+1}, \quad c_t \geq 0,$$

where $\beta \in (0,1)$ is the discount factor, u is the utility function, f is the production function and z_t is a productivity shock.[22] The productivity shock is drawn from a finite set $\hat{\mathcal{Z}} := \{z^1, \ldots, z^n\}$, $n \geq 2$, with z_t governed by a time-homogeneous Markov chain with transition probabilities $p(z, z^+) := \mathbf{P}(z_{t+1} = z^+ \mid z_t = z) > 0$, for all $z, z^+ \in \widehat{\mathcal{Z}}$.[23] We make some standard assumptions on preferences and technology. The utility function $u\colon \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is continuous, strictly increasing, and strictly concave on $\mathbb{R}_+$ (on $\mathbb{R}_{++}$ if $u(0) = -\infty$), with $\lim_{c \downarrow 0} u(c) = u(0)$; it is twice continuously differentiable for $c > 0$ and $\lim_{c \downarrow 0} u'(c) = \infty$. The production function $f\colon \mathbb{R}_+ \times \widehat{\mathcal{Z}} \to \mathbb{R}_+$ is continuously differentiable, strictly increasing and strictly concave in k with $\lim_{k \downarrow 0} f_k(k, z) = \infty$ for all $z \in \widehat{\mathcal{Z}}$ (where f_k denotes $\partial f(k,z)/\partial k$), and is such that there exists a $k^{\max} > 0$ with $f(k, z) < k$ for all $k > k^{\max}$ and all $z \in \widehat{\mathcal{Z}}$. The initial conditions are $k_0 > 0$ and $z_0 \in \widehat{\mathcal{Z}}$ given.

The problem can be set up recursively. Letting k^+ denote next period's capital stock and z^+ next period's shock, the value function satisfies

$$v(k, z) = \max_{0 \leq k^+ \leq f(k,z)} u(f(k, z) - k^+) + \beta \mathbf{E}_{z^+} \left[v(k^+, z^+) \mid z \right] \tag{3.3}$$

where $\mathbf{E}_{z^+}$ is expectation over z^+. Let $k_{t+1} = g(k_t, z_t)$ be the policy function, and $c(k, z) := f(k, z) - g(k, z)$. The following is standard (see, e.g., Stokey et al. (1989, Chapter 10)): $c(k, z)$ and $g(k, z)$ are continuous and increasing in k; moreover $v(k, z)$ is increasing, strictly concave and differentiable in k for $k > 0$.

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where f is the production function, depending on capital k and productivity shock z, and u is the utility function. Since the arguments of the utility function and production function depend on the policy function themselves, this condition is difficult to check a priori, except in special cases. One such special case is where the capital and productivity shock are perfect complements in production, in which case the left-hand-side of the above inequality becomes infinitely large.

[22]For this section we use f to denote the production function and g to denote the policy function.

[23]Brock and Mirman (1972) also assume a finite set of states but assumed the stochastic shock process was i.i.d.

For $k > 0$, the solution to the maximization problem in (3.3) is interior.[24] Thus, the first-order and envelope conditions are given by:

$$u'(c(k,z)) = \beta \mathbf{E}_{z^+}\left[v_k(g(k,z),z^+)) \mid z\right], \tag{3.4}$$

$$v_k(k,z) = u'(c(k,z))f_k(k,z). \tag{3.5}$$

Combining (3.4) and (3.5), we have:

$$v_k(k,z) = \beta f_k(k,z)\mathbf{E}_{z^+}\left[v_k(g(k,z),z^+)) \mid z\right], \tag{3.6}$$

$$u'(c(k,z)) = \beta \mathbf{E}_{z^+}\left[u'(c(g(k,z),z^+))f_k(g(k,z),z^+) \mid z\right]. \tag{3.7}$$

Define the upper and lower envelopes of the policy functions: $\bar{g}(k) := \max_z g(k,z)$ and $\underline{g}(k) := \min_z g(k,z)$. These functions are continuous and increasing and $\bar{g}(k) \geq \underline{g}(k)$. Define $k'' := \inf\{k > 0 \mid \bar{g}(k) \leq k\}$ and $k' := \sup\{0 < k \leq k'' \mid \underline{g}(k) = k\}$. To establish convergence on a positive and bounded interval, $[k', k'']$, we first prove the following lemma (the proof can be found in the Appendix).

LEMMA 2 *(i) There is an $\epsilon > 0$ such that $\underline{g}(k) > k$ for all $k \in (0,\epsilon)$; (ii)If $k'' > k'$ then for all $k > k'$, $\underline{g}(k) < k$.*

The first part of the lemma adapts the arguments of Mitra and Roy (2012) (see also Roy and Zilcha (2012)) to establish that there is growth with probability one near zero capital. That is, the capital stock must optimally increase if capital is close to zero and hence $k', k'' > 0$. This result is derived from the Inada condition on the marginal product at zero and the assumption that transition probabilities are positive. The second part of the lemma ensures that sets above k'' are transient, and allows the corollary to be applied in a straightforward manner. Note that k'' exists by the continuity of $\bar{g}(k)$ and is finite because $\bar{g}(k) \leq f(k,z) < k$ for all $k > k^{\max}$ and z.

Assume that a degenerate stationary equilibrium at $k > 0$ does not exist (see below for some conditions that guarantee this). Then we can establish convergence in the uniform metric of the distributions of k_t to a unique non-degenerate stationary distribution π with support in $[k', k'']$ for any initial value $k_0 > 0$: First, $k'' > k'$ since $k'' = k'$ implies $\underline{g}(k'') = \bar{g}(k'')$ and hence a degenerate steady state at k''. Next, for any $k > 0$ where $k \notin [k', k'']$, it follows from the definitions that all such k are transient and there is a positive probability sequence such that k will transit to this interval. Next, with $\bar{g}(k) > k > \underline{g}(k)$ for all $k \in (k', k'')$ by definition of k' and by part (ii) of the lemma, we can show that the relevant mixing condition of Corollary 1 is satisfied on $[k', k'']$. To see this start from (k'', z_0); repeatedly applying $\underline{g}$ yields the strictly decreasing

[24] We have $k^+ > 0$ because the marginal return to saving, $\beta\mathbf{E}_{z^+}[u'(c^+)f_k(k^+,z^+) \mid z] \to \infty$ as $k^+ \downarrow 0$ which therefore exceeds $u'(c_t)$ for all k^+ near zero. Similarly, the condition $\lim_{c\downarrow 0} u'(c) = \infty$ ensures $k^+ < f(k,z)$.

sequence $(\underline{g}(g(k'', z_0)), \underline{g}^{(2)}(g(k'', z_0)), \ldots)$ where $\underline{g}^{(n)}$ denotes the n-fold composition of $\underline{g}$. It follows that $\lim_{T\to\infty} \underline{g}^{(T)}(g(k'', z_0)) = k'$.[25] Therefore, fixing some $\hat{k} \in (k', k'')$, there exists a finite sequence of productivity shocks $(z_t)_{t=1}^{T}$ with $z_t \in \arg\min_{z\in Z}\{g(\underline{g}^{(t-1)}(g(k'', z_0)), z)\}$ such that the occurrence of $(z_t)_{t=1}^{T-1}$ implies $k_T \leq \hat{k}$. Moreover, the sequence $(z_t)_{t=1}^{T}$, with $z_T = z_0$, has positive probability since all the transition probabilities are positive. By a symmetric argument, using $\bar{g}(k)$ and starting from (k', z_0), there exists a positive probability, finite sequence of productivity shocks $(\widetilde{z}_t)_{t=1}^{\widetilde{T}-1}$ whose occurrence implies $k_{\widetilde{T}} \geq \hat{k}$. Corollary 1 can then be applied with $c = \hat{k}$, $N_1 = \widetilde{T}$, and $N_2 = T$ to establish convergence as claimed.

Under mild conditions degenerate steady states do not exist. Here are two examples:

1. First suppose that preferences are CRRA, $u(c) = c^{1-\alpha}/(1-\alpha)$, $\alpha > 1$, and shocks are multiplicative with $z \in \mathbb{R}_{++}$, $z^1 < z^2 < \ldots z^n$ say, and $f(k, z) = zh(k)$ and write $h'(k) \equiv dh/dk$.

We have $u' f_k = (zh(k) - k)^{-\alpha} zh'(k)$, and

$$\partial (u' f_k) / \partial z = -\frac{h'(k)\left((\alpha-1)h(k)z + k\right)}{(zh(k)-k)^{1+\alpha}} < 0 \tag{3.8}$$

by $c = zh(k) - k > 0$. Consider (3.7) at a degenerate steady state $k > 0$, where $k = g(k, z)$ all z:

$$u'(f(k,z) - k) = \beta \mathbf{E}_{z^+}\left[u'(f(k,z^+) - k) f_k(k, z^+) \mid z\right]. \tag{3.9}$$

We have $\beta f_k(k, z^n) > 1$ since otherwise by $f_k(k, z^n) > f_k(k, z^i)$, for $i < n$, $\beta f_k(k, z^i) < 1$ for $i < n$, and by $u'' < 0$, $u'(f(k, z^1) - k) > u'(f(k, z^i) - k)$ for $i > 1$, so we get

$$u'(f(k, z^1) - k) > \beta f_k(k, z^i) u'(f(k, z^i) - k)$$

all i. This violates (3.9) for $z = z^1$. But then we get

$$\begin{aligned} \beta f_k(k, z^1) u'(f(k, z^1) - k) &> \beta f_k(k, z^2) u'(f(k, z^2) - k) > \ldots \\ &> \beta f_k(k, z^n) u'(f(k, z^n) - k) > u'(f(k, z^n) - k) \end{aligned}$$

where the final inequality follows by $\beta f_k(k, z^n) > 1$ and the rest by (3.8). This implies the RHS of (3.9) exceeds the LHS for $z = z^n$, contradicting optimality.

2. Suppose that in addition to any persistent shock to output, there is also a transitory component to the shock (i.e., such that distribution over future shocks is unaffected by the transitory component); specifically suppose there exist z',

[25] Otherwise, if $\lim_{T\to\infty} \underline{g}(T)(g(k'', z_0)) = \widetilde{k} > k'$, then the continuity of $\underline{g}$ implies $\underline{g}(\widetilde{k}) = \widetilde{k}$, which contradicts $\bar{g}(k) > k > \underline{g}(k)$ for all $k \in (k', k'')$.

$z'' \in \widehat{\mathcal{Z}}$, such that $f(k,z') > f(k,z'')$, $\forall k > 0$, and $p(z',z) = p(z'',z)$ for all $z \in \widehat{\mathcal{Z}}$.

Then the choice of next period's capital stock differs for at least two of the possible realizations of z: Taking states z' and z'' as above where $f(k,z') > f(k,z'')$, it follows that $g(k,z') > g(k,z'')$ for $k > 0$, and hence there cannot be a degenerate steady state.[26]

3.3. *Limited Commitment Risk-Sharing Model*

In this section we consider the inter-temporal risk-sharing model with limited commitment. Kocherlakota (1996) (see also, for example, Alvarez and Jermann, 2000, 2001; Ligon et al., 2002; Thomas and Worrall, 1988) provides a convergence result for the long-run distribution of risk-sharing transfers when shocks to income are finite and i.i.d. His model has two, infinitely-lived, risk averse agents with per-period, strictly concave and differentiable utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ defined over consumption, and a common discount factor β. Agent 1 has a random endowment $y_t > 0$ at date $t = 0, 1, \ldots$, and agent 2 has a random endowment $Y - y_t > 0$ where $Y > 0$ is a constant aggregate income. The endowment shock is drawn from a finite set $\mathcal{Y} := \{y^1, \ldots, y^n\}$, $n \geq 2$, with y_t governed by a Markov chain with stationary transition probabilities $p(y, y^+) := \mathbf{P}(y_{t+1} = y^+ \mid y_t = y) > 0$, for all $y, y^+ \in \mathcal{Y}$. There is no credit market but agents can transfer income between themselves at any date. Although Kocherlakota (1996) assumes the endowment shocks are i.i.d., we will show that this convergence result is easily extended to the case where y_t is a Markov chain. It is important to consider this non-i.i.d. case. The inter-temporal risk-sharing model with limited commitment has been most frequently applied to village economies where income is predominantly derived from farming. Farm incomes are often found to be to be positively serially correlated.[27]

To study optimal risk sharing in this limited commitment context, let $h^t = (y_0, y_1, \ldots, y_t)$ denote the history of income realizations, agents choose a sequence of history-dependent transfers $X_t(h^t)$ from agent 1 to agent 2 subject to $-Y + y_t \leq X_t(h^t) \leq y_t$ for each h^t and the self-enforcing constraints that neither agent prefers autarky from that point on after any history over the agreed transfer plan.

[26]Suppose otherwise, that $g(k,z') \leq g(k,z'')$. It follows from $f(k,z') > f(k,z'')$ that $c(k,z') > c(k,z'')$. This leads to a contradiction of (3.4). The LHS of (3.4) is strictly lower at z' than at z'' by the concavity of the utility function. Conversely, by the concavity of v, $v_k(g(k,z'),z^+) \geq v_k(g(k,z''),z^+)$ at each z^+, meaning the the RHS of (3.4) is no lower at at z' than at z'', by the assumption that $p(z',z^+) = p(z'',z^+)$ all z^+.

[27]For example, Bold and Broer (2016) use the ICRISAT data of three Indian villages and find estimated autocorrelation coefficients of around $0.61 - 0.77$.

In particular, the self-enforcing constraints for the two agents are

$$\begin{aligned} u(y_t - X_t(h^t)) + \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(y_{t+s} - X_t(h^{t+s}))] \\ \geq u(y_t) + \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(y_{t+s}))], \\ u(Y - y_t + X_t(h^t)) + \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(Y - y_{t+s} + X_t(h^{t+s}))] \\ \geq u(Y - y_t) + \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(Y - y_{t+s}))], \end{aligned}$$

for each date t and h^t. An *efficient risk-sharing arrangement* will solve (for some feasible U^0):

$$\max_{\{X_t\}} \mathbf{E}_0[\sum_{s=0}^{\infty} \beta^s u(y_s - X_s(h^s))] \quad \text{s.t.} \quad \mathbf{E}_0[\sum_{s=0}^{\infty} \beta^s u(Y - y_s + X_s(h^s))] \geq U^0.$$

and subject to the self-enforcing constraints. It is well known (see, e.g., Ligon et al., 2002) that the solution at each date has the following property: For each realization y, there is a time-invariant interval $I_y = [\underline{c}_y, \overline{c}_y]$, $\underline{c}_y \leq \overline{c}_y$, such that

$$c_{t+1}\left(h^{t+1}\right) := y_{t+1} - X_{t+1}(h^{t+1}) = \begin{cases} \overline{c}_{y_{t+1}} & \text{if } c_t(h^t) > \overline{c}_{y_{t+1}} \\ c_t(h^t) & \text{if } c_t(h^t) \in I_{y_{t+1}} \\ \underline{c}_{y_{t+1}} & \text{if } c_t(h^t) < \underline{c}_{y_{t+1}} \end{cases},$$

and there is a one-to-one correspondence between feasible U^0 and agent 1's initial consumption $c_0(h^0) \in [\underline{c}_{y_0}, \overline{c}_{y_0}]$. We can write this in the form (2.4) as $c_{t+1} = f(c_t, z_t)$ where $z_t := y_{t+1}$, and where

$$f(c, z) = \begin{cases} \overline{c}_z & \text{if } c > \overline{c}_z \\ c & \text{if } c \in I_z \\ \underline{c}_z & \text{if } c < \underline{c}_z \end{cases}.$$

The function $f(c, z)$ is clearly monotone increasing in c. If $f(c, z)$ were also increasing in z and the Markov process determining y were persistent, then the approach of Hopenhayn and Prescott (1992) could be used. However, even if the Markov process determining y is monotone, the dependence of $f(c, z)$ on z is not easy to derive from the primitives of the model because $\overline{c}_z$ and $\underline{c}_z$ are computed as part of the optimal solution. They are determined by the slopes of the value functions of the dynamic programming problem and depend on all elements of the problem.[28]

The first-best risk-sharing allocation is sustainable for some U^0 if and only if $\cap_z I_z \neq \emptyset$. Kocherlakota (1996) shows (his Proposition 4.2) that if shocks are i.i.d. and if the first-best is not sustainable then the distribution of transfers converges weakly to the same non-degenerate distribution for all U^0. We now show how to easily extend this result to the case where shocks follow a Markov chain without making assumptions on the monotonicity of $f(c,z)$ in z. Define $c_{\min} := \min_z \overline{c}_z$, $c_{\max} := \max_z \underline{c}_z$. If the first-best is not sustainable, $\cap_z I_z = \emptyset$, then $c_{\min} < c_{\max}$. If $c_t \in [c_{\min}, c_{\max}]$, $c_{t+1} = f(c_t, z_t) \in [c_{\min}, c_{\max}]$ for all z_t. Define $c := (c_{\min} + c_{\max})/2$. Using the notation of Corollary 1 (where $[c_{\min}, c_{\max}]$ replaces $[\alpha, \beta]$), let $N_1 = N_2 = 2$, $z_{1,1} \in \arg\max_z \underline{c}_z$, $z_{1,2} \in \arg\min_z \overline{c}_z$. For any z_0, all the assumptions of the corollary are satisfied. Thus, there exists a unique distribution π such that the distributions of c_t converge to π in the uniform metric for any initial value $c_0 \in [c_{\min}, c_{\max}]$. Clearly, $c_t \in [\underline{c}_{z_0}, \overline{c}_{z_0}] \cup [c_{\min}, c_{\max}]$ all t, and $[\underline{c}_{z_0}, \overline{c}_{z_0}] \backslash [c_{\min}, c_{\max}]$ is transient.

If the first-best is sustainable, then the mixing condition is not satisfied. In that case it can be seen immediately that there is monotone convergence to a first-best allocation (the limit allocation is dependent on the initial condition).

4. CONCLUSION

In this paper we have established convergence results that can be used in a range of models whose dynamics can be represented by a stochastic recursion, and which satisfy two main conditions; first, for a given value of the exogenous driving process, the future value of the endogenous variable is monotone increasing in its current value; secondly, the driving process is regenerative. The latter includes as a special case irreducible finite Markov chains. These two conditions, along with a standard mixing condition, guarantee weak convergence to a unique stationary distribution.

This extends the existing results on convergence of monotone Markov processes that assume the driving process is i.i.d. or assume that the driving process is itself a monotone Markov process (Hopenhayn and Prescott, 1992). This extension is important because most economic models take the driving process for the underlying shocks to be exogenous and therefore it is useful to have results for a broader class of stochastic driving processes. Moreover, we do not require that the stochastic recursion is monotone in the second argument. This is particularly useful when the stochastic recursion is derived as a policy function of a dynamic programming problem because establishing monotonicity in the shock process might require extra restrictions on preferences or technology.

[28] One case where it is known that monotonicity in z can be established is if one of the agents is risk-neutral. This is the case studied by Thomas and Worrall (1988). We are unaware of any results on the monotonicity in z in other more general cases. Fortunately, our method does not rely on establishing such monotonicity properties and can also be applied if the income process were negatively autocorrelated.

We have applied our approach to three workhorse models in macroeconomics extending our understanding of stability in these models. Our Theorem 2 and its corollary can also be readily used to establish convergence to a unique stationary distribution for any monotone stochastic recursion in a regenerative environment where the appropriate mixing condition is satisfied.

APPENDIX

Proof of Theorem 2

PROOF: Define a sequence $Y_{n+1} = X_{T_n}$ for all $n \geq 0$. This sequence is clearly a Markov chain and can therefore be represented in the form

$$Y_{n+1} = g(Y_n, \eta_n)$$

with an i.i.d. driving sequence

$$\eta_n = \left(\tau_n, Z_{T_{n-1}}, .., Z_{T_n - 1}\right)$$

and where the function g is defined by

$$g(Y_n, \eta_n) = f^{(\tau_n)}\left(Y_n, Z_{T_{n-1}}, .., Z_{T_n-1}\right).$$

In addition, this recursion is again monotone in the first argument, due to the monotonicity of function f. The assumptions of the theorem imply that there exists $c \in [a, b]$ such that

$$\mathbf{P}(Y_1 \leq c | Y_0 = b) = \mathbf{P}\left(\widetilde{X}^{(b)}_{T_1 - T_0} \leq c\right) > 0$$

and

$$\mathbf{P}(Y_1 \geq c | Y_0 = a) = \mathbf{P}\left(\widetilde{X}^{(a)}_{T_1 - T_0} \geq c\right) > 0.$$

Hence, the assumptions of Theorem 1 are satisfied with the same c and with $N = 1$. This implies the first statement of the theorem.

We prove the second statement now. For any t, let $\nu(t)$ be such that $T_{\nu(t)} \leq t < T_{\nu(t)+1}$, so t belongs to the $(\nu(t)+1)$st cycle. Let $\psi_t = (t - T_{\nu(t)}, Z_{T_{\nu(t)}}, \ldots, Z_{t-1})$ and denote $\psi_{t,1} = t - T_{\nu(t)}$ and $\psi_{t,2} = (Z_{T_{\nu(t)}}, \ldots, Z_{t-1})$, so $\psi_t = (\psi_{t,1}, \psi_{t,2})$. For any fixed $k > 0$ and for all sufficiently large t, consider a vector of random vectors[29] $(\eta_{\nu(t)-k}, \eta_{\nu(t)-k+1}, \ldots, \eta_{\nu(t)}, \psi_t)$. By the classical result on regenerative processes (see, e.g., Asmussen (2003)), for any fixed $k > 0$ and as t tends to infinity, the joint distribution of random vectors $(\eta_{\nu(t)-k}, \eta_{\nu(t)-k+1}, \ldots, \eta_{\nu(t)}, \psi_t)$

[29] Note that each such vector is the sequence of shocks, together with lengths, of each of the previous $k + 1$ completed cycles plus shocks and length of the incomplete cycle up to time t.

converges in the total variation norm to the limiting distribution of a vector of random vectors, say, $(\eta^{-k}, \dots, \eta^0, \psi^0)$:

$$\delta_{t,k} := \sup_B |\mathbf{P}((\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)}, \psi_t) \in B) - \mathbf{P}((\eta^{-k}, \dots, \eta^0, \psi^0) \in B)| \to 0$$

as $t \to \infty$.

Random vectors $\eta^{-k}, \dots, \eta^0, \psi^0$ are mutually independent, and each of the η^{-j}, $j = 0, \dots, k$, has the distribution of the "typical cycle", while random vector ψ^0 represents the left half of the "integrated cycle", and its first coordinate ψ_1^0 has the integrated tail distribution $\mathbf{P}(\psi_1^0 = l) = \frac{1}{\mathbf{E}\tau_1}\mathbf{P}(\tau_1 > l)$, for $l = 0, 1, \dots$. In what follows, we use representation $\psi^0 = (\psi_1^0, \psi_2^0)$ where ψ_2^0 is the rest of vector ψ^0 (and, in particular, it is l-dimensional if $\psi_1^0 = l$).

Further, a more advanced construction is possible: one can introduce (on a common probability space with all earlier defined random variables) a stationary sequence $(\eta_t^{-k}, \dots, \eta_t^0, \psi_t^0)$ such that

$$\mathbf{P}(A_{t,k}) = \delta_{t,k},$$

where we denote

$$A_{t,k} = \{(\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)}, \psi_t) \neq (\eta_t^{-k}, \dots, \eta_t^0, \psi_t^0)\}$$

(see, e.g., Chapter 1 in Lindvall (2002)).[30] In the rest of the proof, we assume such a coupling to be given.

Introduce $\widehat{Y}_{t,0}^k = \widetilde{Y}_{t,0}^k = Y_{\nu(t)-k}$ and

$$\widehat{Y}_{t,m+1}^k = g(\widehat{Y}_{t,m}^k, \eta_{\nu(t)-k+m}), \quad m = 0, .., k-1$$

and

$$\widetilde{Y}_{t,m+1}^k = g(\widetilde{Y}_{t,m}^k, \eta_t^{-k+m}), \quad m = 0, .., k-1.$$

Consider now

$$\mathbf{P}\left(\widehat{Y}_{t,k}^k \neq \widetilde{Y}_{t,k}^k\right) = \mathbf{P}\left(g^{(k)}(Y_{\nu(t)-k}, (\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)})) \neq (g^{(k)}(Y_{\nu(t)-k}, (\eta_t^{-k}, \dots, \eta_t^0))\right)$$
$$\leq \mathbf{P}(\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)}) \neq (\eta_t^{-k}, \dots, \eta_t^0)) \leq \mathbf{P}(A_{t,k}) = \delta_{t,k},$$

with the obvious notation for $g^{(k)}$.

Introduce also Z_0^k as a random variable with distribution π and independent of $(\eta^{-k}, \dots, \eta^0, \psi^0)$ and let

$$Z_{t,m+1}^k = g(Z_{t,m}^k, \eta_t^{-k+m}), \quad m = 0, .., k-1.$$

[30]In applied probability, such a construction is frequently called a "successful coupling of transient and stationary sequences".

Note that $Z^k_{t,m}$ has distribution π for all t, k and m.

Due to the first statement of the theorem, we have that, as $k \to \infty$, the distribution of the random variable $\widehat{Y}^k_{t,k}$ converges to distribution π in the total variation norm and, hence, in the uniform metric. We can therefore, for any $\varepsilon > 0$, choose k such that, for any r,

$$\left|\mathbf{P}(\widehat{Y}^k_{t,k} \le r) - \mathbf{P}(Z^k_{t,k} \le r)\right| \le \varepsilon$$

and then

$$\begin{aligned}
&\left|\mathbf{P}\left(\widetilde{Y}^k_{t,k} \le r\right) - \mathbf{P}\left(Z^k_{t,k} \le r\right)\right| \\
&= \left|\mathbf{P}\left(\widetilde{Y}^k_{t,k} \le r, \widehat{Y}^k_{t,k} = \widetilde{Y}^k_{t,k}\right) + \mathbf{P}\left(\widetilde{Y}^k_{t,k} \le r, \widehat{Y}^k_{t,k} \ne \widetilde{Y}^k_{t,k}\right) - \mathbf{P}\left(Z^k_{t,k} \le r\right)\right| \\
&= \left|\mathbf{P}\left(\widehat{Y}^k_{t,k} \le r, \widehat{Y}^k_{t,k} = \widetilde{Y}^k_{t,k}\right) + \mathbf{P}\left(\widetilde{Y}^k_{t,k} \le r, \widehat{Y}^k_{t,k} \ne \widetilde{Y}^k_{t,k}\right) - \mathbf{P}\left(Z^k_{t,k} \le r\right)\right| \\
&= \left|\mathbf{P}\left(\widehat{Y}^k_{t,k} \le r\right) - \mathbf{P}\left(\widehat{Y}^k_{t,k} \le r, \widehat{Y}^k_{t,k} \ne \widetilde{Y}^k_{t,k}\right) + \mathbf{P}\left(\widetilde{Y}^k_{t,k} \le r, \widehat{Y}^k_{t,k} \ne \widetilde{Y}^k_{t,k}\right) - \mathbf{P}\left(Z^k_{t,k} \le r\right)\right| \\
&\quad \le \left|\mathbf{P}\left(\widehat{Y}^k_{t,k} \le r\right) - \mathbf{P}\left(Z^k_{t,k} \le r\right)\right| + 2\mathbf{P}\left(\widehat{Y}^k_{t,k} \ne \widetilde{Y}^k_{t,k}\right) \le 2\delta_{t,k} + \varepsilon.
\end{aligned}$$

Now, using similar arguments, for any r and any $l = 0, 1, \ldots,$

$$\begin{aligned}
&\left|\mathbf{P}\left(X_t \le r, t - T_{\nu(t)} = l\right) - \mathbf{P}\left(f^{(l)}(\widetilde{X}^{(\pi)}_0, \psi^0_{0,2}) \le r, \psi^0_{0,1} = l\right)\right| \\
&= \left|\mathbf{P}\left(X_t \le r, t - T_{\nu(t)} = l\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, \psi^0_{t,2}) \le r, \psi^0_{t,1} = l\right)\right| \\
&= \left|\mathbf{P}\left(f^{(l)}(\widehat{Y}^k_{t,k}, \psi_{t,2}) \le r, t - T_{\nu(t)} = l\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, \psi^0_{t,2}) \le r, \psi^0_{t,1} = l\right)\right| \\
&\le \left|\mathbf{P}\left(f^{(l)}(\widetilde{Y}^k_{t,k}, \psi^0_{t,2}) \le r, \psi^0_{t,1} = l\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, \psi^0_{t,2}) \le r, \psi^0_{t,1} = l\right)\right| + 2\delta_{t,k}.
\end{aligned}$$

Note that for any $l = 1, 2, \ldots$ and any $v \in \mathcal{Z}^l$, the set $S_l(v, r) = \{x \,:\, f^{(l)}(x, v) \le r\}$ is an interval of the form $[a, b)$ or $[a, b]$, for some b. Therefore,

$$\begin{aligned}
&\left|\mathbf{P}\left(f^{(l)}(\widetilde{Y}^k_{t,k}, \psi^0_{t,2}) \le r, \psi^0_{t,1} = l\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, \psi^0_{t,2}) \le r, \psi^0_{t,1} = l\right)\right| \\
&= \mathbf{P}(\psi^0_{t,1} = l) \int \left|\mathbf{P}\left(f^{(l)}(\widetilde{Y}^k_{t,k}, v) \le r\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, v) \le r\right)\right| \mathbf{P}(\psi^0_{t,2} \in dv \mid \psi^0_{t,1} = l) \\
&= \mathbf{P}(\psi^0_{t,1} = l) \int \left|\mathbf{P}\left(\widetilde{Y}^k_{t,k} \in S_l(v, r)\right) - \mathbf{P}\left(Z^k_{t,k} \in S_l(v, r)\right)\right| \mathbf{P}(\psi^0_{t,2} \in dv \mid \psi^0_{t,1} = l) \\
&\qquad\qquad \le \mathbf{P}(\psi^0_{t,1} = l) \sup_w |\mathbf{P}(\widetilde{Y}_{t,k} \le w) - \mathbf{P}(Z^k_{t,k} \le w)|.
\end{aligned}$$

Thus,

$$\left|\mathbf{P}\left(X_t \le r, t - T_{\nu(t)} = l\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, \psi^0_{t,2}) \le r, \psi^0_{t,1} = l\right)\right|$$

tends to 0, and the same holds for any finite sum in l. From the general theory of renewal processes (see, e.g., Asmussen (2003)) it is known that the family of random variables $\{t - T_{\nu(t)}\}$ is tight. Recall that this means that

$$\Delta(l) := \sup_t \mathbf{P}(t - T_{\nu(t)} > l) \to 0$$

as $l \to \infty$. Therefore, for any $\varepsilon > 0$, one can choose $L > 0$ such that $\Delta(L) + \mathbf{P}(\psi^0_{t,1} > L) \le \varepsilon$ for any t. Then

$$\begin{aligned}&\left|\mathbf{P}\left(X_t \le r\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, \psi^0_{t,2}) \le r\right)\right| \\ &\le \sum_{l=0}^{L} \left|\mathbf{P}\left(X_t \le r, t - T_{\nu(t)} = l\right) - \mathbf{P}\left(f^{(l)}(Z^k_{t,k}, \psi^0_{t,2}) \le r\psi^0_{t,1} = l\right)\right| + \varepsilon \to \varepsilon,\end{aligned}$$

as $t \to \infty$. Letting ε go to zero, we arrive at the second statement of the theorem.

The proof of the convergence of (X_t, Z_t) follows the exact same lines, with an extra event added in each of the probabilities. We omit this derivation as the formulae are rather cumbersome but do not contain any additional technical difficulties. *Q.E.D.*

Proof of Corollary 1

PROOF: We have to show that Corollary 1 follows from Theorem 2. For that, we have to define a typical (say, first) regenerative cycle and show that all the conditions of Theorem 2 hold. Assume that $Z_0 = z_0$, so $T_0 = 0$. Let $T_1 = \tau_1 = \min\{t > 0 : Z_t = z_0\}$, then the aperiodicity means that $G.C.D.\{t : \mathbf{P}(T_1 = t) > 0\} = 1$. Let $T_n = \sum_1^n \tau_j$ where τ_j are i.i.d. copies of τ_1. Let the conditions of the Corollary hold, and k_i be the number of occurrences of z_0 in the sequence $z_{j,i}$, for $i = 1, 2$. Let L be the *least common multiple* of k_1 and k_2,

$$L = \min\{l \ : \ l/k_1 \ \text{ and } \ l/k_2 \ \text{ are integers}\}.$$

Let α be a random variable that takes values 0 and 1 with equal probabilities and does not depend on any of the processes defined in the model. Then define a regenerative cycle as follows: $\widehat{T}_0 = 0$ and

$$\widehat{T}_1 = T_1\alpha + T_L(1 - \alpha).$$

That is, we suppose that our regenerative cycle is either a single cycle or a sum of L cycles, with equal probabilities. Then all the conditions of Theorem 2 hold (with $\widehat{T}_i$ in place of T_i). Indeed, condition (2.2) follows since it holds for τ_1, and since $\widehat{T}_1$ is not bigger than T_L, the sum of L copies of τ_1. Condition (2.3) follows because the set of all t such that $\mathbf{P}(\widehat{T}_1 = t) > 0$ includes the set of all t such that $\mathbf{P}(\tau_1 = t) > 0$ and, therefore,

$$G.C.D.\{t \ : \ \mathbf{P}(\widehat{T}_1 = t) > 0\} \le G.C.D.\{t \ : \ \mathbf{P}(\tau_1 = t) > 0\},$$

so, given aperiodicity, both greatest common divisors are equal to 1. Finally, ε_1 in (2.9) is not smaller than $\frac{1}{2}p_1\delta_1 > 0$ and, similarly, ε_2 in (2.10) is not smaller than $\frac{1}{2}p_2\delta_2 > 0$. *Q.E.D.*

Proof of Lemma 1.

PROOF: Define

$$\hat{c} := (\bar{e} - \underline{e})/(1 - (\beta R)^{1/\gamma}).$$

Clearly, there exists $\hat{x}$ such that for $x > \hat{x}$, $c(x,e) > \hat{c}$ for all $e \in E$.[31] Suppose that, at some (x, e) with $x > \hat{x}$, $f(x,e) \geq x$. We demonstrate a contradiction. Since $f(x,e) > \underline{x}$, the Euler condition holds with equality:

$$u'(c(x,e)) = \beta R \mathbf{E}_{e^+}\left[u'(c(f(x,e),e^+)) \mid e\right]. \tag{A.1}$$

(A.1) implies that there exists $X^+ \in \mathcal{E}$ with $Q(e, X^+) > 0$ and such that $u'(c(x,e)) \leq \beta R u'(c(f(x,e),e^+))$ for $e^+ \in X^+$. Thus for $e^+ \in X^+$,

$$c(f(x,e),e^+)^{-\gamma} \geq (\beta R)^{-1} c(x,e)^{-\gamma},$$

so

$$c(f(x,e),e^+) \leq (\beta R)^{1/\gamma} c(x,e). \tag{A.2}$$

By $c(x,e) > \hat{c}$, we have from (A.2):

$$c(x,e) - c(f(x,e),e^+) \geq (1 - (\beta R)^{1/\gamma}) c(x,e) \tag{A.3}$$

$$> (\bar{e} - \underline{e}). \tag{A.4}$$

Then

$$\begin{aligned} f(f(x,e),e^+) &= R(f(x,e) + e^+ - c(f(x,e),e^+)) \\ &> R(x + e^+ + (\bar{e} - \underline{e}) - c(x,e)) \\ &\geq R(x + e - c(x,e)) \\ &= f(x,e), \end{aligned} \tag{A.5}$$

where the first line follows from the budget constraint, the second from $f(x,e) \geq x$ and (A.4), the third from $e^+ \geq \underline{e}$ and $\bar{e} \geq e$, and the last from the budget constraint. Defining $x_t = x$, $x_{t+1} = f(x,e)$, $x_{t+2} = f(f(x,e),e^+)$ etc., we can express (A.5) as $x_{t+2} \geq x_{t+1}$. Repeating the logic of (A.2) and (A.5), starting at $(f(x,e),e^+)$ for some $e^+ \in X^+$ there is some $X^{++} \in \mathcal{E}$ with $Q(e^+, X^{++}) > 0$ at $t+2$ such that $x_{t+3} > x_{t+2}$ and such that

$$c_{t+2} \leq (\beta R)^{2/\gamma} c(x,e),$$

etc. Iterating, we get eventually that $c_{t+n} < \hat{c}$ while $x_{t+n} > \hat{x}$, a contradiction. *Q.E.D.*

[31] For $a \geq (R/(R-1))(1-\beta)^{1/(1-\gamma)}\hat{c}$ setting $c_t = ((R-1)/R)a + e_t$ all t (so that a_t is constant at a) yields a discounted utility greater than $\hat{c}^{1-\gamma}/(1-\gamma)$; this is higher utility than any policy with $c(a, e_t) \leq \hat{c}$ which yields at most $\hat{c}^{1-\gamma}/(1-\gamma)$.

Details of convergence result in Section 3.1

Assume the initial state (at time $t = 0$) e_0 is the atom of the chain and suppose that $x_0 = \hat{x}$. Maximum consumption at $t = 0$ if all resources are used is $\overline{c} := \hat{x} - \underline{x}/R + e_0$. We can also define a lower bound on consumption at any date by $\underline{c} > 0$.[32] Choose $T \in \{1, 2, \dots\}$ and $\xi > 0$ so that

$$\text{(A.6)} \qquad \overline{c}^{-\gamma} > (\beta R)^T \underline{c}^{-\gamma} + \xi.$$

(This implies that the agent would like, if feasible, to transfer a small amount of consumption forward from T periods ahead.) Suppose that $\Pr[x_t = \underline{x}] = 0$ for all $t > 0$. We shall establish a contradiction. For any $\Delta > 0$, we can choose $\varepsilon > 0$ so that $\mathbf{P}(x_t < \underline{x} + \varepsilon$ for at least one $t \in \{1, \dots, T\}) < \Delta$ (using the right continuity of the distribution of x_t, say F_t, with the hypothesis that $F_t(\underline{x}) = 0$ for $t \leq T$, choose ε so that at each t, $F_t(\underline{x} + \varepsilon) < \Delta/T$). It follows that an increase in consumption at $t = 0$ of amount $\lambda \leq \varepsilon R^{-T}$ can be financed by a reduction at date T (but otherwise keeping time t consumption c_t, $1 \leq t < T$, at its original level), i.e., $x_t \geq \underline{x}$ for $1 \leq t \leq T$, with probability at least $(1 - \Delta)$ since assets at t would be $x_t - R^t \lambda \geq x_t - \varepsilon \geq \underline{x}$ for $t \leq T$. To a first-order, the discounted utility cost is at most $\lambda(\beta R)^T \underline{c}^{-\gamma}$. Otherwise reduce c_t to restore assets to x_t when the credit constraint first binds at $t < T$, at a cost of at most $\lambda \underline{c}^{-\gamma}$. The change in utility to a first order is thus at least

$$\lambda \left(\overline{c}^{-\gamma} - \Delta \underline{c}^{-\gamma} - (1 - \Delta)(\beta R)^T \underline{c}^{-\gamma} \right).$$

Choosing Δ small so that $\Delta \underline{c}^{-\gamma} < \xi$, the term multiplying λ is positive, using (A.6), and so for λ small (so that $\lambda \leq \varepsilon R^{-T}$ is satisfied, where ε depends on Δ, and that higher order terms are small enough) there is a profitable deviation. Hence $\hat{t} := \min\{t > 0 : \mathbf{P}(x_t = \underline{x}) > 0\} < \infty$.

Next, define times as $T_0 = \min\{t \geq 0 : e_t = e_0\}$, and for $j = 0, 1, \dots,$

$$T_{j+1} = \min\left\{t \geq T_j + \hat{t} : e_t = e_0\right\}.$$

Thus the sequence $\{e_t\}$ with associated times $\{T_n\}$ is regenerative and satisfies (2.2)-(2.3). Moreover consider the process $\tilde{x}_t^{(\alpha)}$starting at $t = 0$ from α and satisfying recursion $\tilde{x}_{t+1}^{(\alpha)} = f(\tilde{x}_t^{(\alpha)}, e_{T_0+t})$. By the above, $\{\tilde{x}_{\hat{t}}^{(\hat{x})} = \underline{x}\}$ has positive probability. Now consider $\tilde{x}_{T_1-T_0}^{(\underline{x})}$. By the monotonicity of f in its first argument, $\tilde{x}_t^{(\underline{x})} \leq \tilde{x}_t^{(\hat{x})}$ for all t, and if $\tilde{x}_{\hat{t}}^{(\hat{x})} = \underline{x}$, then also $\tilde{x}_{\hat{t}}^{(\underline{x})} = \underline{x}$, so $\tilde{x}_t^{(\underline{x})} = \tilde{x}_t^{(\hat{x})}$ for $t \geq \hat{t}$ and conditional on hitting $\underline{x}$ at $\hat{t}$, $\tilde{x}_{\hat{t}}^{(\hat{x})}$ and $\tilde{x}_{\hat{t}}^{(\underline{x})}$ coincide at each $t \geq \hat{t}$. Consequently c exists satisfying conditions (2.9)-(2.10) of Theorem 2 and the result follows.

[32]Since $c_t = ((R-1)/R)\underline{x} + \bar{e} > 0$ is always feasible (by assumption on $\underline{x}$), this implies a lower bound to utility; consumption below some positive level implies a discounted utility below this bound.

Proof of Lemma 2.

PROOF: (i) Suppose that $\underline{g}(k) \leq k$. Consider z_τ, k_τ such that $k_{\tau+1} = \underline{g}(k_\tau) = g(k_\tau, z_\tau)$, that is consider the shock that depletes capital at the maximum rate. Let $\phi(k) = \inf_z f_k(k, z)$ be the greatest lower bound on the marginal product as a function of k. We have

$$\begin{aligned}
u'(c(k_\tau, z_\tau)) &= \beta \mathbf{E}_{z_{\tau+1}} \left[u'(c(g(k_\tau, z_\tau), z_{\tau+1})) f_k(g(k_\tau, z_\tau), z_{\tau+1}) \mid z_\tau \right] \\
&= \beta \mathbf{E}_{z_{\tau+1}} \left[u'(c(\underline{g}(k_\tau), z_{\tau+1})) f_k(\underline{g}(k_\tau), z_{\tau+1}) \mid z_\tau \right] \\
&\geq \beta \phi(k_\tau) \mathbf{E}_{z_{\tau+1}} \left[u'(c(\underline{g}(k_\tau), z_{\tau+1})) \mid z_\tau \right] \\
&\geq \beta \phi(k_\tau) \mathbf{E}_{z_{\tau+1}} \left[u'(c(k_\tau, z_{\tau+1})) \mid z_\tau \right].
\end{aligned}$$

The first equality follows by equation (3.7). The second equality follows by the definition $k_{\tau+1} = \underline{g}(k_\tau)$. The inequality in the third line follows by $\underline{g}(k_\tau) \leq k_\tau$ and the definition of ϕ, and the final inequality follows by $\underline{g}(k_\tau) \leq k_\tau$ and $c(k, z)$ increasing in k. Since $z_\tau = z^i$ for some state i, the above inequality (deleting terms for states $j \neq i$) implies

$$u'(c(k_\tau, z^i)) \geq \beta \phi(k_\tau) u'(c(k_\tau, z^i)) p(z^i, z^i).$$

Since $u'(c) > 0$, it therefore follows that $1 \geq \beta \phi(k_\tau) p(z^i, z^i)$. Let $\rho := \min_i p(z^i, z^i)$. By assumption $\rho > 0$, and therefore $\phi(k_\tau) \leq 1/(\beta\rho)$ for all k_τ. Since $\beta > 0$ and $\rho > 0$, equivalently, $k_\tau \geq \phi^{-1}(1/(\beta\rho))$. Letting $\epsilon = \phi^{-1}(1/(\beta\rho))$, the assumption that $f_k(k, z) \to \infty$ for all z as $k \downarrow 0$ implies $\epsilon > 0$ and hence we have $k_\tau \geq \epsilon$ for all τ. Thus, it follows that $g(k_\tau, z_\tau) > k_\tau$ for all $k_\tau < \epsilon$ and all z_τ. (ii) Suppose not. Then by continuity of $\underline{g}(k)$, $\exists\ \hat{k} > k'$ such that $\underline{g}(\hat{k}) = \hat{k}$. By definition of k' and assumption that $k' < k''$, $\underline{g}(k'') < \bar{g}(k'')$ $(= k'')$ and so $\hat{k} > k''$. Consider any $(k, z) \in [k', k''] \times \widehat{\mathcal{Z}}$. Then $g(k, z) \in [k', k'']$ since $g(k, z) \geq \underline{g}(k) \geq \underline{g}(k') = k'$ where the second inequality follows from g increasing in k, and the equality from the definition of k'; likewise, $g(k, z) \leq \bar{g}(k) \leq \bar{g}(k'') = k''$ where the second inequality follows from g increasing in k, and the equality from the definition of k''. Similarly, for $k \geq \hat{k}$, $g(k, z) \geq \hat{k}$, $\forall z \in \widehat{\mathcal{Z}}$, since $g(k, z) \geq \underline{g}(k) \geq \underline{g}(\hat{k}) = \hat{k}$. We shall demonstrate a contradiction. Take any $(\bar{k}, z_0) \in (k', k'') \times \widehat{\mathcal{Z}}$, and define recursively

$$\begin{aligned}
&\bar{k}_0 = \bar{k}; \\
&\bar{k}_\tau = g(\bar{k}_{\tau-1}, z_{\tau-1}) \qquad \tau = 1, \ldots, N.
\end{aligned} \tag{A.7}$$

Iterating (3.6) $N > 0$ times:

$$v_k(\bar{k}, z) = \mathbf{E} \left[\beta^N \Pi_{\tau=0}^{N-1} f_k(\bar{k}_\tau, z_\tau) v_k(\bar{k}_N, z_N) \mid z_0 \right]. \tag{A.8}$$

Likewise, for any $\tilde{k} \geq \hat{k}$, defining $\tilde{k}_\tau$ (analogously to $\bar{k}_\tau$) starting from $(\tilde{k}, z_0)$,

$$v_k(\tilde{k}, z) = \mathbf{E} \left[\beta^N \Pi_{\tau=0}^{N-1} f_k(\tilde{k}_\tau, z_\tau) v_k(\tilde{k}_N, z_N) \mid z_0 \right]. \tag{A.9}$$

By $\bar{k}_\tau \in [k', k'']$, $\tilde{k}_\tau \geq \hat{k}$, $k'' < \hat{k}$, and the the strict concavity of f and v in k :

$$\text{(A.10)} \qquad f_k(\bar{k}_\tau, z_\tau) \geq \gamma f_k(\tilde{k}_\tau, z_\tau) \qquad a.s.,$$

for some $\gamma > 1$, and

$$\text{(A.11)} \qquad v_k(\bar{k}_N, z_N) > v_k(\tilde{k}_N, z_N) \qquad a.s.$$

Thus, from (A.8), (A.9), (A.10) and (A.11):

$$v_k(\bar{k}, z) > \gamma^N v_k(\tilde{k}; z).$$

Since $v_k(\bar{k}; z) < \infty$ by $\bar{k} > 0$, $\gamma > 1$, $v_k(\tilde{k}; z) > 0$, letting $N \to \infty$ yields a contradiction. *Q.E.D.*

Acknowledgement: The research of S. Foss was partially supported by EPSRC grant EP/I017054/1 and by RSF grant 17-11-01173. The research of S. Shneer was supported by EPSRC grant EP/L026767/1. The Research of J. Thomas and T. Worrall was supported by ESRC grant ES/L009633/1.

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