STOCHASTIC STABILITY OF MONOTONE ECONOMIES IN REGENERATIVE ENVIRONMENTS

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February 2016

Abstract
We introduce and analyze a new class of monotone stochastic recursions in a regenerative environment which is essentially broader than that of Markov chains. We prove stability theorems and apply our results to three canonical models in recursive economics, generalizing some known stability results to the cases when driving sequences are not independent and identically distributed. We also revisit the case of monotone Markovian models (or, equivalently, stochastic recursions with i.i.d. drivers) and provide a simplified version of the proof of a stability result given previously by Dubins and Freedman (1966) and Bhattacharya and Majumdar (2007)

Keywords: Monotone Economy, Markov Chain, Stochastic Recursion, Driving Sequence, Renegerative Sequence, Existence and Uniqueness of a Stationary Distribution, Stochastic Stability, Bewley-Huggett-Aiyagari Model, Risk-Sharing Model.
JEL Classification: C61, C62.

1. introduction

The dynamic evolution of a number of economic models can be described by a *stochastic recursive sequence (SRS)*, or *stochastic recursion* of the form (see, e.g., Stachurski, 2009, ch.6)

$$X_{t+1} = f(X_t, \xi_t) \quad \text{a.s.}, \tag{1}$$

where $\{\xi_t\}$ is a stochastic process with $\xi_t \in \mathcal{S}$, $X \in \mathcal{X}$ is the state variable of economic interest and $f : \mathcal{X} \times \mathcal{S} \to \mathcal{X}$ is an appropriately measurable function. The process $\{\xi_t\}$ is known as the *driving sequence* of the stochastic recursion. For a given X_0 and given (random) values of $\xi_0, \ldots, \xi_{t-1}$, the system (1) generates a (random) value of X_t.

It is well-known (see, e.g., Borovkov and Foss, 1992) that, under extremely general conditions (see Section 2.1 for details), any time-homogeneous Markov chain may be represented as an SRS (1) with independent and identically distributed (i.i.d.) driving elements $\xi_0, \xi_1, \ldots$.

The questions we address in this paper are whether and under what conditions there exists an equilibrium distribution for X and if so, whether it is unique; whether the sequence X_t converges and if it does, whether the long-run distribution is independent of the initial conditions.

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The answers to these questions depend on the spaces $\mathcal{X}$ and $\mathcal{S}$, the function f and the nature of the driving sequence. In this paper we are concerned with the case where the function f is monotone *increasing* in X and ξ is a *regenerative process*. We make appropriate assumptions on $\mathcal{X}$ and $\mathcal{S}$ that are specified below. Loosely, a stochastic process is regenerative if it can be split into independent and identically distributed (i.i.d.) cycles. That is, if there exists a subsequence of (random) dates such that the process has the same probabilistic behavior between any two consecutive dates in the subsequence. The cycle lengths (lengths of time intervals between these dates) may also be random, in general, with the only requirement that they have a finite mean value. As an example, consider a finite-state time-homogeneous Markov chain with a single closed class of communicating states. If the chain starts in some state z_0, then the subsequence of dates corresponds to the dates at which the chain revisits state z_0. Between each of these dates the chain has the same probabilistic behavior.[1] The class of regenerative processes is large and includes not only ergodic Markov chains, but also renewal processes, Brownian motion, waiting times in general queues and so on.[2]

Before explaining our approach in more detail, we outline three traditional approaches that are used to address stability and uniqueness issues for SRS of the type described by equation (1). First, when $\{\xi_t\}$ is i.i.d., the process for X_t is Markov and standard existence and convergence results for Markov processes can be applied. For example, when f is monotone in the first argument, it is well-known that there is convergence to a unique invariant distribution if a mixing or splitting condition holds (see, e.g., Bhattacharya and Majumdar, 2007; Hopenhayn and Prescott, 1992; Stokey et al., 1989).[3]

Second, stability results are also known in a more general setting where the driving sequence $\{\xi_t\}$ is stationary or even asymptotically stationary (this literature originated with Loynes (1962), see, e.g., Borovkov and Foss (1992) and references therein). By *stationarity* we mean *stationarity in the strong sense*, that is, for any finite k, the distribution of a finite-dimensional vector $(\xi_t, \dots, \xi_{t+k})$ does not depend on t. The most basic result is that if the state space for the X's is partially ordered and possesses least element, say 0, and if SRS $X_{t+1} = f(X_t, \xi_t)$ starts from the bottom point $X_0 = 0$, with f monotone increasing in the first

[1] An i.i.d. process is one that is regenerative at every date.

[2] We are not the first to consider regenerative processes in the economics literature. For example, Kamihigashi and Stachurski (2015) consider perfect simulation of a stochastic recursion of the form $X_{t+1} = f(X_t, \xi_t)\mathbb{1}\{X_t \geq x\} + \epsilon_t \mathbb{1}\{X_t < x\}$ where $\mathcal{X} = [a, b]$, $x \in (a, b)$, f is increasing in X and $\{\xi_t\}$ and $\{\epsilon_t\}$ are i.i.d. The process regenerates for values $X_t < x$. This process arises in models of industry dynamics with entry and exit (see Hopenhayn, 1992). It is a Markov process, but it is not monotone unless the distribution of $f(x, \xi)$ stochastically dominates the distribution of ϵ.

[3] Stokey et al. (1989) use the Feller property, which is a continuity requirement, together with monotonicity and a mixing condition to derive the results. Hopenhayn and Prescott (1992) develop an existence result using monotonicity alone, and combined with a mixing condition, establish that uniqueness and stability follow.

argument, then the distribution of X_t is monotone increasing in t and, given that the sequence is *tight*,[4] it converges to a limit which is the *minimal* stationary solution to recursion (1.1). In general, there may be many solutions, and for the minimal solution to be unique, one has to require existence of so-called *renovating events* (for details see, e.g., Brandt, 1985; Foss, 1983). These results seem to have been relatively little used in the economics literature although in Bewley (1987) it is assumed that there is a Markov driving sequence for shocks that starts from a stationary state.

A third situation where results are known is considered by Stokey et al. (1989, Chapter 9) and Hopenhayn and Prescott (1992). If $\{\xi_t\}$ is itself a Markov chain, or equivalently an SRS of the form $\xi_t = g(\xi_{t-1}, \varepsilon_{t-1})$ with i.i.d. $\{\varepsilon_t\}$, then $Y_t = (X_t, \xi_t)$ is a time-homogeneous Markov chain, equivalently, an SRS of the form $Y_{t+1} = F(Y_t, \varepsilon_t) := (f(X_t, \xi_t), g(\xi_t, \varepsilon_t))$. Then, provided a mixing condition is satisfied, one can use the monotone convergence approach to establish convergence of the extended Markov chain Y_t. This is the approach generally used in the economics literature. There are however, three disadvantages to this approach. First, in order to apply monotone convergence results, it is required that function g is increasing in the first argument. That is, it is required that the driving process is itself monotone (positively correlated). Whilst this may be natural in many economic contexts, it may be restrictive in others.[5] Second, to apply monotone convergence results, it is required that function f is monotone (increasing) in both arguments, not just the first argument. This can be problematic in situations where the SRS is derived as a policy function of a dynamic programming problem. In this case, establishing monotonicity in the second argument may require extra restrictions on preferences and technology. This is the case in the one sector stochastic optimal growth model with correlated shocks that is studied by Donaldson and Mehra (1983) and others; see section 3.2. Third, the fact that the state space for the extended state variable, $\mathcal{X} \times \mathcal{S}$, is of a larger dimension, may create additional technical difficulties and establishing that the mixing condition is satisfied may become less straightforward.

In this paper we exploit the i.i.d. cycle property of regenerative processes. We use this property to construct a Markov process defined at the regeneration times driven by an i.i.d. random variable. Together with an analogue of the monotone mixing or *splitting condition* of Bhattacharya and Majumdar (1999, condition (1.2)) this can be used to establish convergence to a unique stationary distribution.

We develop our approach in a simple scenario with a compact and completely ordered state space $\mathcal{X}$ (which may be taken to be $[0, 1]$ without loss of generality). In the case where the driving sequence is i.i.d., the splitting condition says that (we focus here on the i.i.d. case for simplicity of notation and explanations),

[4] Tightness in this context means that for any $\varepsilon > 0$ there exists K_ε such that $\mathbf{P}(X_t \geq K_\varepsilon) \leq \varepsilon$ for all t.

[5] For example, if the states that the driving process represents have no natural ordering, there may be no reordering of states such that the process is monotone.

for some $c \in [0,1]$, there is a finite time N such that for the Markov chain $X_t^{(1)}$ that starts from the maximal state $X_0^{(1)} = 1$ at time zero (with any ξ_0), the probability $\mathbf{P}(X_N^{(1)} \leq c) > 0$ and, second, for the Markov chain $X_t^{(0)}$ that starts from the minimal state $X_0^{(0)} = 0$ at time zero (with any ξ_0), the probability $\mathbf{P}(X_N^{(0)} \geq c) > 0$. In Section 2.1 we reproduce a result of Bhattacharya and Majumdar (2007) for the case where f is monotone increasing, the driving sequence is i.i.d. and the splitting condition holds (our Theorem 1) that shows there is exponentially fast convergence to a unique stationary distribution. Theorem 2 in Section 2.2 extends this result to allow for a regenerative driving sequence. A corollary to this theorem (Corollary 1) is provided in Section 2.3 that considers the important special case where the driving sequence is itself an aperiodic Markov chain with a positive atom.[6] For such regenerative driving sequences, our approach generalizes the standard result whilst avoiding the disadvantages mentioned above. In particular, we establish convergence to a unique stationary distribution without needing to assume the driving process is itself monotone or that the function f is increasing in the second argument. In addition, our convergence applies directly to the state space of interest, $\mathcal{X}$, rather than the extended state space $\mathcal{X} \times \mathcal{S}$.

We apply our approach to three important workhorse models in macroeconomics. These are the Bewley-Imrohoroglu-Huggett-Aiyagari precautionary savings model of Bewley (1987), İmrohoroğlu (1992), Huggett (1993) and Aiyagari (1994), the one-sector stochastic optimal growth model of Brock and Mirman (1972), and the risk-sharing under limited commitment model of Kocherlakota (1996). In each of these examples, we are able to demonstrate uniqueness and stability results under less restrictive assumptions than the current literature.

The paper is organized as follows. In Section 2, we describe the model and provide our main results. First, we describe regenerative processes. Next, we review the results of Bhattacharya and Majumdar (2007) for an i.i.d. driving sequence. Then, we present the main results showing that if a mixing condition similar to that given in Bhattacharya and Majumdar (2007) are satisfied between the dates when the driving sequence regenerates, then stability holds. Section 3 presents the three applications of our main result. The proofs of the main result is given in the text and subsidiary proofs are put in the Appendix.

2. THE MAIN MODEL

In this section, we outline the main properties of regenerative processes, provide some examples of regeneration for Markov chains, and finally introduce our main model, which is a stochastic recursive sequence with a regenerative driver.

[6] All the economic examples we consider here are of this type. However, our main theorem does apply more generally to any regenerative process and this result may be useful for other applications.

Let $Z_t, t \geq 0$ be a (one-sided) regenerative sequence on a general measurable space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$. The sequence is *regenerative* if there exists an increasing sequence of integer-valued random variables (times) $0 = T_{-1} \leq T_0 < T_1 < T_2 < \ldots$ such that, for $\tau_n = T_n - T_{n-1}, n \geq 0$, the vectors

(2) $$\{\tau_n, Z_{T_{n-1}}, \ldots, Z_{T_n - 1}\}$$

are independent for $n \geq 0$ and identically distributed for $n \geq 1$. A random vector (2) is called a *cycle* with *cycle length* τ_n.

In this paper, we deal with discrete time and, therefore, assume T_n to be integer-valued. Furthermore, we assume that

(3) $$\mathbf{E}\tau_1 < \infty.$$

It is known (see, e.g., Asmussen, 2003) that if, in addition, regenerative times are *aperiodic*,

(4) $$G.C.D.\{n : \mathbf{P}(\tau_1 = n) > 0\} = 1,$$

then Z_t has a unique stationary/limiting distribution, say π, and converges to it in the total variation norm:

$$\sup_{B \in \mathcal{B}_{\mathcal{Z}}} |\mathbf{P}(Z_t \in B) - \pi(B)| \to 0, \quad \text{a.s.} \quad t \to \infty.$$

Example 1 A typical regenerative scenario is produced by a positive recurrent time-homogeneous Markov chain with an atom (*Markov chain with a positive atom*, for short). This is a Markov chain with a general state space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ that contains a point $z_0 \in \mathcal{Z}$ such that, for any $z \in \mathcal{Z}$,

$$T_1^z = \min\{t \ : Z_t = z_0 \mid Z_0 = z\} < \infty \quad \text{a.s.}$$

and

$$\mathbf{E}T_1^{z_0} < \infty.$$

Here are two more examples, of a "non-Markovian" regeneration for a Markov chain and of a regenerative non-Markovian model.[7]

Example 2 Assume that a Markov chain $\{Z_t\}$ takes values in a finite state space $\mathcal{Z}$ and is irreducible and aperiodic. Let $p(z_i, z_j) = \mathbf{P}(Z_1 = z_j | Z_0 = z_i)$ be the transition probabilities of the Markov chain. Take $K \geq 2$ and consider a sequence $(\omega_1, \omega_2, .., \omega_K) \in \mathcal{Z}^K$. If $p(\omega_i, \omega_{i+1}) > 0$ for each $i = 1, .., K-1$, then we say that this sequence forms a *word*. Let

$$T_0 \equiv T_0(\omega_1, \omega_2, .., \omega_K) = \min\{t \geq K : Z_{t-K+1} = \omega_1, Z_{t-K+2} = \omega_2, ..Z_t = \omega_K\}$$

[7] Any discrete-space regenerative process may be made Markov, by extending the state space. But the cost is that the state space may be impractically large. Example 3 below is therefore for illustrative purposes.

be the first time the word $(\omega_1, \omega_2, .., \omega_K)$ appears in the sequence $\{Z_t\}$. Similarly, for each $j = 1, 2, \ldots$, let

$$T_j = \min\{t > T_j + K : Z_{t-K+1} = \omega_1, Z_{t-K+2} = \omega_2, ..Z_t = \omega_K\}$$

be the second, third, etc. times the same word appears in the sequence. Clearly, $\{Z_t\}$ is a regenerative process with regenerative times $\{T_j\}$.

Suppose now there are $I \geq 2$ words $(\omega_{i,1}, \omega_{i,2}, .., \omega_{i,K_i})$ with corresponding lengths K_i, $i = 1, .., I$, such that $\omega_{1,K_1} = \omega_{2,K_2} = ... = \omega_{I,K_I}$ (i.e. they all end with the same character). Assume that none of these words is a "sub-word" of any other (i.e. cannot be obtained from another word by removing a number of characters at the beginning and/or at the end). Let

$$T_0 = \min_{i=1,..I}\{T_0(\omega_{i,1}, \omega_{i,2}, .., \omega_{i,K_i})\}.$$

One can say that here T_0 is the first appearance time of any of the words.

Let $K = \max_{i=1,\ldots,I} K_i$ and define times $T_1, T_2, \ldots$ by induction: given T_j, we let

$$T_{j+1} = \min\{t \geq T_j + K : (Z_{t-K_i+1}, \ldots, Z_t) = (\omega_{i,1}, \ldots, \omega_{i,K_i}) \quad \text{for some} \quad i = 1, \ldots, I\}.$$

Then again $\{T_j\}$ forms a sequence of regenerative times for Markov chain $\{Z_t\}$.

Example 3 Let the state space $\mathcal{Z}$ contain only two states, $\mathcal{Z} = \{0, 1\}$. Consider the following two sequences, $V_1 = (1, 0, 0)$ and $V_2 = (1, 1, 1, 0)$. We start from $Z_0 = 0$. Assume a regenerative cycle takes values V_1 and V_2 with equal probabilities. This means the following. Let $\{T_1, T_2, \ldots\}$ be a sequence of i.i.d. random variables taking values 3 and 4 with equal probabilities, $\mathbf{P}(T_i = 3) = \mathbf{P}(T_i = 4) = 1/2$.

We let the length of the first cycle be T_1. If $T_1 = 3$, we let $Z_1 = 1$ and $Z_2 = Z_3 = 0$, and if $T_1 = 4$, we let $Z_1 = Z_2 = Z_3 = 1$ and $Z_4 = 0$. Then the length of the second cycle is T_2, and if $T_2 = 3$, then $Z_{T_1+1} = 1$ and $Z_{T_1+2} = Z_{T_1+3} = 0$, and if $T_2 = 4$, then $Z_{T_1+1} = Z_{T_1+2} = Z_{T_1+3} = 1$ and $Z_{T_1+4} = 0$. By induction, we define the process $\{Z_t\}$ for all t, with letting, for any $k = 1, 2, \ldots$,

$$Z_{T_1+\ldots+T_k+1} = 1 \quad \text{and} \quad Z_{T_1+\ldots+T_k+2} = Z_{T_1+\ldots+T_k+3} = 0$$

if $T_{k+1} = 3$, and

$$Z_{T_1+\ldots+T_k+1} = Z_{T_1+\ldots+T_k+2} = Z_{T_1+\ldots+T_k+3} = 1 \quad \text{and} \quad Z_{T_1+\ldots+T_k+4} = 0$$

if $T_{k+1} = 4$.

Now we turn to the general framework. Assume that, along with regenerative process $\{Z_t\}$, we are given a family of random variables $\{\xi_t^z\}_{z\in\mathcal{Z},-\infty<t<\infty}$ that take values in a measurable space $(\mathcal{V}, \mathcal{B}_\mathcal{V})$. We assume that

- this family does not depend on $\{Z_t\}$,
- contains mutually independent random variables,
- for each $z \in \mathcal{Z}$, random variables $\{\xi_t^z\}_{t\geq 1}$ are i.i.d. with common distribution G_z, and that,
- for each y, the function $G_z(y)$ is measurable with respect to z.

The main aim of the paper is to study the behavior of a recursive sequence

$$X_{t+1} = f\left(X_t, \xi_t^{Z_t}\right) \tag{5}$$

assuming that
- the function f is measurable and is *monotone* in the first argument, with respect to some ordering;
- sequence $\{Z_t\}$ is regenerative and satisfies conditions (3)-(4).

2.1. *I.i.d. driving sequence*

We start with a particular case when $Z_t = z = const$ for each t. We then drop the upper index and simply write

$$X_{t+1} = f\left(X_t, \xi_t\right), \tag{6}$$

where the ξ's are i.i.d. We revisit some results from Bhattacharya and Majumdar (2007) (see also Dubins and Freedman, 1966).

The relation between time-homogeneous Markov chains (with a general measurable state space $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ and recursions (6) with i.i.d. drivers is well-understood (see, e.g., Borovkov and Foss, 1992; Kifer, 1986): if the sigma-algebra $\mathcal{B}_{\mathcal{X}}$ is countably generated, then a Markov chain may be represented as a stochastic recursion (6) with an i.i.d. driving sequence $\{\xi_t\}$. In particular, any real-valued or vector-valued time-homogeneous Markov chain may be represented as a stochastic recursion (6).

In what follows, we restrict our attention to real-valued X_t and, moreover, assume that

$$\text{the state space} \quad \mathcal{X} \quad \text{is the closed interval} \quad [0,1].$$

Introduce the *uniform distance* between probability distributions on the real line as

$$d(F,G) = \sup_x |F(x) - G(x)|.$$

Here $F(x) = F(-\infty, x]$ and $G(x) = G(-\infty, x]$ are the distribution functions. Let $F(x-) = F(-\infty, x)$ and $G(x-) = G(-\infty, x)$. Then, clearly,

$$d(F,G) = \sup_x |F(x-) - G(x-)| \equiv \sup_x \max\left(|F(x-) - G(x-)|, |F(x) - G(x)|\right)$$

Next, we assume the function f to be *monotone increasing* in the first argument: for each $v \in \mathcal{V}$ and for each $0 \leq x_1 \leq x_2 \leq 1$,

$$f(x_1, v) \leq f(x_2, v).$$

Then, in particular, for any $v \in \mathcal{V}$ and any $y \in [0,1]$, the set

$$S(v, y) := \{x \ : \ f(x, v) \leq y\}$$

is an interval containing 0 (i.e. an interval of the form $[0, a)$ or $[0, a]$). In particular, it is a closed interval if f is continuous in x.

Similarly, the sets

$$S^{(2)}(v_1, v_2, y) := \{x \ : \ f(f(x, v_1), v_2) \leq y\}$$

also form intervals containing zero, for all $v_1, v_2 \in \mathcal{V}$ and for all $y \in [0,1]$. Furthermore, by the induction argument, we define similarly sets $S^{(t)}(\overline{v}, y)$ for all $y \in [0,1]$ and for all vectors $\overline{v} = (v_1, .., v_t)$ with $v_i \in \mathcal{V}$ for all i.

We write for short

$$\mathbf{P}^{(x)}(\cdot) = \mathbf{P}(\cdot \mid X_0 = x).$$

We also denote by $F_t^{(x)}$ the distribution function of the random variable X_t if $X_0 = x$. More generally, we denote by $F_t^{(\mu)}$ the distribution function of X_t if X_0 has distribution μ.

Theorem 1 *(see Bhattacharya and Majumdar, 2007, Theorem 3.5.1 for a slightly more general version).*
Assume that time-homogeneous Markov chain X_t is represented by the stochastic recursion (6) *with i.i.d. driving sequence $\{\xi_t\}$, where function $f : [0,1] \times \mathcal{V} \to [0,1]$ is monotone increasing in the first argument.*
Assume there exists a number $c \in [0,1]$ and integer $N \geq 1$ such that

$$\varepsilon_1 := \mathbf{P}^{(1)}(X_N \leq c) > 0$$

and

$$\varepsilon_2 := \mathbf{P}^{(0)}(X_N \geq c) > 0.$$

Then, there exists a distribution π on $[0,1]$ such that

$$\sup_x d(F_t^{(x)}, \pi) \to 0, \quad t \to \infty \tag{7}$$

exponentially fast.
Furthermore, π is the unique stationary distribution for the Markov chain X_t.

In order to make the paper self-contained a concise proof of Theorem 1 that is useful for proving our main result (Theorem 2) is given in the Appendix. The first results in this direction were obtained in Dubins and Freedman (1966) (under an additional assumption of continuity of the mapping f).

Remark 1 Theorem 1 is easily generalized to a case where the set $\mathcal{S}$ has a partial order, $\leq$, such that there exists least element $s_0 \in \mathcal{S}$ and greatest element $s_1 \in \mathcal{S}$ and f is monotone increasing in the first argument (with respect to the partial order $\leq$). In this case, the mixing condition requires that there exists an $\varepsilon > 0$, an integer $N \geq 1$ and sets $\mathcal{C}_u \subset \mathcal{S}$ and $\mathcal{C}_l \subset \mathcal{S}$ such that for every element $s \in \mathcal{S}$, there either exists an element $c \in \mathcal{C}_u$ such that $s \geq c$, or there exists an element $c \in \mathcal{C}_l$ such that $s \leq c$; and for every $c \in \mathcal{C}_u$, $\mathbf{P}^{(s_1)}(X_N \leq c) > \varepsilon$, and for every $c \in \mathcal{C}_l$, $\mathbf{P}^{(s_0)}(X_N \geq c) > \varepsilon$.

Remark 2 Note that the assumptions of Theorem 1 may be rewritten in an equivalent form as follows: assuming in addition that the trajectories $X_t^{(1)}$ and $X_t^{(0)}$ are mutually independent, there is a positive N such that $\mathbf{P}(X_N^{(1)} \leq X_N^{(0)}) \geq \delta > 0$. This condition is called a *strong reversing condition* by Kamihigashi and Stachurski (2014) because then, due to the monotonicity, it also holds for any other pair of initial conditions $0 \leq x_0 < y_0 \leq 1$, with the same δ and N, namely $\mathbf{P}(X_N^{(y_0)} \leq X_N^{(x_0)}) \geq \delta > 0$. One can consider our approach in a more general setting (developed for Markov chains by Kamihigashi and Stachurski (2014) and Szeidl (2013)), assuming, more generally, that the state space may not contain top and/or bottom points (then the δ and N may, in general, depend on (x_0, y_0)) and, moreover, that the order is only partial. In particular, Szeidl (2013) suggested a reasonable "replacement", say, for a top point (if one does not exist) by a random "top" point. Namely, assume, say, the state space for the Markov chain is the positive half-line $[0, \infty)$ where there is the minimal element 0 but there is no maximal element, and suppose that a Markov chain X_t is defined by a stochastic recursion $X_{t+1} = f(X_t, \xi_t)$ with i.i.d. $\{\xi_t\}$. Assume that there exists a random measure μ on $[0, \infty)$ such that if $X_0 \sim \mu$ and if X_0 does not depend on ξ_0, then $X_1 = f(X_0, \xi_0)$ is *stochastically smaller* than X_0 (this means $\mathbf{P}(X_1 \leq x) \geq \mathbf{P}(X_0 \leq x)$, for all x). Then the distribution μ may play a role of a new random "top" point if, for example, the distribution of μ has an unbounded support. For instance, if there exists another function, say h such that $f(x, y) \leq h(x, y)$ for all x, y and that a Markov chain $Y_{t+1} = h(Y_t, \xi_t)$ admits a unique stationary distribution, say μ. If μ can be easily found/determined, it may play the role of a random "top" point.

Here is a simple example. Assume that X_t is a discrete-time birth-and-death-process with immigration at 0, i.e. a non-negative integer-valued Markov chain, which is homogeneous in time and with transition probabilities $\mathbf{P}(X_1 = 1 \mid X_0 = 0) = 1 - \mathbf{P}(X_1 = 0 \mid X_0 = 0) = p_0 > 0$ and, for $k = 1, 2, \ldots$, let $\mathbf{P}(X_1 = k + 1 \mid X_0 = k) = 1 - \mathbf{P}(X_1 = k - 1 \mid X_0 = k) = p_k$. Furthermore assume that the p_k are non-decreasing in k (this makes the Markov chain monotone), that all are smaller than $1/2$ and, moreover, that $\lim_{k \to \infty} p_k = p < 1/2$. Consider a Markov chain Y_t with transition probabilities $\mathbf{P}(Y_1 = k + 1 \mid Y_0 = k)) = p = 1 - \mathbf{P}(Y_1 = \max(0, k - 1)) \mid Y_0 = k)$. Then this Markov chain has a unique stationary distribution μ (which is clearly geometric), and it gives a random "top" point.

In Borovkov and Foss (1992), a similar concept of a stationary majorant was developed and studied, where the top sequence $\{X_t\}$ is assumed to be stationary.

Similar ideas have been developed earlier in the area of so-called "perfect simulation", with introducing an artificial random "top" point (see, e.g., (Foss and Tweedie , 1998) and (Corcoran and Tweedie, 2001) and the references therein) for simulation "from the past".

2.2. *Regenerative driving process*

We now turn our attention to the general regenerative setting (5), but continue to assume that the state space $\mathcal{X} = [0,1]$. We formulate and prove a general result and then give some important corollaries and examples.

We introduce an auxiliary process $\widetilde{X}_t^{(a)}$ that starts from $\widetilde{X}_0^{(a)} = a$ at time 0, and follows the recursion

$$\widetilde{X}_{t+1}^{(a)} = f\left(\widetilde{X}_t^{(a)}, \xi_t^{Z_{T_0+t}}\right) \quad \text{for all} \quad t \geq 0.$$

Remark 3 The auxiliary process $\widetilde{X}_t^{(a)}$ coincides in distribution with the process X started at time T_0 from the state a, and assumptions (8) and (9) ensure the mixing (similar to that guaranteed by conditions of Theorem 1) over a typical cycle (from T_0 to T_1) of the regenerative process Z.

More generally, we consider an auxiliary process $\widetilde{X}_t^{(F)}$ that follows the recursion

$$\widetilde{X}_{t+1}^{(F)} = f\left(\widetilde{X}_t^{(F)}, \xi_t^{Z_{T_0+t}}\right) \quad \text{for all} \quad t \geq 0$$

and that starts from a random variable $\widetilde{X}_0^{(F)}$ that has distribution F (and which does not depend on random variables $\{Z_{T_0+t}, \xi_t^{(z)}, z \in \mathcal{Z}, t \geq 0\}$). Denote by $f^{(k)}$ the k-th iteration of function f, so $f^{(1)} = f$ and, for, $k \geq 1$,

$$f^{(k+1)}(x, u_1, \ldots, u_{k+1}) = f\left(f^{(k)}(x, u_1, \ldots, u_k), u_{k+1}\right).$$

Let $f^{(0)}$ be the identity function.

Theorem 2 *Assume that recursive sequence* $\{X_t\}$ *is defined by* (5) *where the function* f *is monotone increasing in the first argument and the sequence* $\{Z_t\}$ *is regenerative with regenerative times* $\{T_n\}$ *that satisfy conditions* (3)-(4).

Assume that the following assumptions hold:

$$\varepsilon_1 := \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(1)} \leq c\right) > 0, \tag{8}$$

and

$$\varepsilon_2 := \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(0)} \geq c\right) > 0. \tag{9}$$

Then there exists a distribution π on $[0,1]$ such that

$$\rho_t := \sup_x d(G_n^{(x)}, \pi) = \sup_x \sup_r |G_n^{(x)}(r) - \pi(-\infty, r]| \to 0, \quad n \to \infty$$

exponentially fast. Here $G_n^{(x)}$ is the distribution of X_{T_n} if $X_{T_0} = x$.

Furthermore, the distributions of X_t converge to distribution

$$\mu(\cdot) = \frac{1}{\mathbf{E}(\tau_1)} \sum_{k=0}^{\infty} \mathbf{P}\left(\tau_1 > k, f^{(k)}\left(\widetilde{X}_0^{(\pi)}, \xi_0^{Z_{T_0}}, \ldots, \xi_{k-1}^{Z_{T_0+k-1}}\right) \in \cdot\right),$$

for any initial value X_0, again in the uniform metric d.

Remark 4 Note that, as in the Markovian case of Theorem 1, we do not require that the function f be continuous in the first argument.

Proof: Introduce a sequence $Y_{n-1} = X_{T_n}$ for all $n \geq 0$. This sequence is clearly a Markov chain and can therefore be represented in the form (6)

$$Y_{n+1} = g(Y_n, \eta_n)$$

with an i.i.d. driving sequence

$$\eta_n = \left(\tau_n, \xi_{T_{n-1}}^{Z_{T_{n-1}}}, .., \xi_{T_n-1}^{Z_{T_n-1}}\right)$$

and the function g is defined by

$$g(Y_n, \eta_n) = f^{(\tau_n)}\left(Y_n, \xi_{T_{n-1}}^{Z_{T_{n-1}}}, .., \xi_{T_n-1}^{Z_{T_n-1}}\right).$$

In addition, this recursion is again monotone in the first argument, due to the monotonicity of function f.

The assumptions of the theorem imply that there exists $c \in [0,1]$ such that

$$\mathbf{P}(Y_1 \leq c | Y_0 = 1) = \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(1)} \leq c\right) > 0$$

and

$$\mathbf{P}(Y_1 \geq c | Y_0 = 0) = \mathbf{P}\left(\widetilde{X}_{T_1-T_0}^{(0)} \geq c\right) > 0.$$

Hence, the assumptions of Theorem 1 are satisfied with the same c and with $N = 1$. This implies the first statement of the theorem.

We prove the second statement now.

For any t, let $\nu(t)$ be such that $T_{\nu(t)} \leq t < T_{\nu(t)+1}$. Let $\psi_t = (t - \nu(t), \xi_{\nu(t)}^{Z_{\nu(t)}}, \ldots, \xi_{t-1}^{Z_{t-1}})$ and denote $\psi_{t,1} = t - \nu(t)$ and $\psi_{t,2} = (\xi_{\nu(t)}^{Z_{\nu(t)}}, \ldots, \xi_{t-1}^{Z_{t-1}})$, so $\psi_t = (\psi_{t,1}, \psi_{t,2})$.

By the classical result on regenerative processes (see e.g., Asmussen (2003)), for any fixed $k > 0$ and as t tends to infinity, the joint distribution of random vectors $(\eta_{\nu(t)-k}, \eta_{\nu(t)-k+1}, \dots, \eta_{\nu(t)}, \psi_t)$ converges in the total variation norm D to the limiting distribution of, say, random vectors $(\eta^{-k}, \dots, \eta^0, \psi^0)$ that are mutually independent:

$$D_{t,k} := \sup_B |\mathbf{P}((\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)}, \psi_t) \in B) - \mathbf{P}((\eta^{-k}, \dots, \eta^0, \psi^0) \in B)| \to 0.$$

In addition, each of the η^{-j}, $j = 0, \dots, k$, has the distribution of the "typical cycle", while random vector ψ^0 represents the left half of the "integrated cycle", and its first coordinate ψ_1^0 has the integrated tail distribution $\mathbf{P}(\psi_1^0 = l) = \frac{1}{\mathbf{E}\tau_1}\mathbf{P}(\tau_1 > l)$, for $l = 0, 1, \dots$. In what follows, we use representation $\psi^0 = (\psi_1^0, \psi_2^0)$ where ψ_2^0 is the rest of vector ψ^0 (and, in particular, it is l-dimensional if $\psi_1^0 = l$).

Furthermore, for any fixed k and t, there is a coupling of $(\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)}, \psi_t)$ and of $(\eta^{-k}, \dots, \eta^0, \psi^0)$ such that

$$\mathbf{P}((\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)}, \psi_t) \neq (\eta^{-k}, \dots, \eta^0, \psi^0)) = D_{t,k}$$

(see, e.g. (Lindvall, 2002, Chapter 1)). In the rest of the proof, we consider a $(\eta^{-k}, \dots, \eta^0, \psi^0)$ where there is such a coupling.

Introduce $\widehat{Y}_{t,0} = Y_{\nu(t)-k}$ and

$$\widehat{Y}_{t,m+1} = g(\widehat{Y}_{t,m}, \eta_{\nu(t)-k+m}), \quad m = 0, .., k-1$$

and

$$\widetilde{Y}_{t,m+1} = g(\widehat{Y}_{t,m}, \eta^{\nu(t)-k+m}), \quad m = 0, .., k-1.$$

Consider now

$$\begin{aligned} &|\mathbf{P}(\widehat{Y}_{t,k} \le r) - \mathbf{P}(\widetilde{Y}_{t,k} \le r)| \\ &= |\mathbf{P}(g^{(k)}(\widehat{Y}_{t,0}, (\eta_{\nu(t)-k}, \dots, \eta_{\nu(t)})) \le r) - \mathbf{P}(g^{(k)}(\widehat{Y}_{t,0}, (\eta^{-k}, \dots, \eta^0)) \le r)| \le D_{t,k} \end{aligned}$$

for any r, with the obvious notation for $g^{(k)}$. This means that the distributions of $\widehat{Y}_{t,k}$ and $\widetilde{Y}_{t,k}$ are close in the uniform metric d. Due to the first statement of the theorem, we also have that, as $k \to \infty$, the distribution of the random variable $\widetilde{Y}_{t,k}$ converges to distribution π in the total variation norm and, hence, in the uniform metric. We can therefore, for any $\varepsilon > 0$, choose k such that, for any r,

$$|\mathbf{P}(\widehat{Y}_{t,k} \le r) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le r)| \le \varepsilon$$

and

$$|\mathbf{P}(\widetilde{Y}_{t,k} \le r) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le r)| \le D_{t,k} + \varepsilon.$$

Now, for any r and any $l = 0, 1, \ldots$,

$$
\begin{aligned}
&|\mathbf{P}(X_t \le r, t-\nu(t)=l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le r, \psi_1^0 = l| \\
&= |\mathbf{P}(f^{(l)}(\widehat{Y}_{t,k}, \psi_{n,2}) \le r, t-\nu(t)=l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le r, \psi_1^0 = l| \\
&\le |\mathbf{P}(f^{(l)}(\widehat{Y}_{t,k}, \psi_{t,2}) \le r, t-\nu(t)=l) - \mathbf{P}(f^{(l)}(\widehat{Y}_{t,k}, \psi_2^0) \le r, \psi_1^0 = l)| \\
&\qquad + |\mathbf{P}(f^{(l)}(\widehat{Y}_{t,k}, \psi_2^0) \le r, \psi_1^0 = l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le r, \psi_1^0 = l|.
\end{aligned}
$$

The first summand does not exceed $\mathbf{P}(t - \nu_t \ne \psi_0^1)$ and therefore converges to zero (uniformly in r). Now turn to the second summand. For any $l = 1, 2, \ldots$ and any $v \in \mathcal{V}^l$, the set $S_l(v,r) = \{x \ : \ f^{(l)}(x,v) \le r\}$ is an interval of the form $[0,a)$ or $[0,a]$, for some a. Therefore,

$$
\begin{aligned}
& |\mathbf{P}(f^{(l)}(\widehat{Y}_{n,k}, v) \le r) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, v) \le r)| \\
= \; & |\mathbf{P}(\widehat{Y}_{t,k} \in S_l(v,r)) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \in S_l(v,r))| \\
\le \; & \sup_w |\mathbf{P}(\widehat{Y}_{t,k} \le w) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le w)|.
\end{aligned}
$$

Then the second summand in the earlier inequality is not bigger than

$$
\begin{aligned}
& \mathbf{P}(\psi_1^0 = l) \int |\mathbf{P}(f^{(l)}(\widehat{Y}_{t,k}, v) \le r) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, v) \le r)| \mathbf{P}(\psi^0 \in dv \mid \psi_1^0 = l) \\
\le \; & \mathbf{P}(\psi_1^0 = l) |\mathbf{P}(\widehat{Y}_{t,k} \le w) - \mathbf{P}(\widetilde{X}_0^{(\pi)} \le w)|.
\end{aligned}
$$

Thus,

$$
\sup_r |\mathbf{P}(X_t \le r, t-\nu(t)=l) - \mathbf{P}(f^{(l)}(\widetilde{X}_0^{(\pi)}, \psi_2^0) \le r, \psi_1^0 = l)|
$$

tends to 0, and the same holds for any finite sum in l.

Since the family of distributions of random variables $t - \nu(t)$ is tight, the second statement of the theorem follows. *Q.E.D.*

Remark 5 In general, we require only the first moment of τ_1 to be finite, so convergence in the regeneration theorem may be as slow as one wishes, and the same holds for convergence of the distribution F_t of random variable X_t to μ. However, if τ_1 has finite $(1+r)$-th moment, then $d(F_t, \mu)$ decays no slower than t^{-r}; and if τ_1 has finite exponential moment, then the convergence is exponentially fast.

Example 4 Consider a very simple toy example, with only two states of environment $\mathcal{Z} = \{1, 2\}$ and with four-state space $\mathcal{X} = \{0, 1, 2, 3\}$ (we prefer to deal with integers, so we re-scale $[0,1]$ to $[0,3]$). Assume sequence $\{Z_t\}$ to be regenerative, with the typical cycle taking two values, $(2,1)$ and $(2,2,1)$, with equal probabilities $1/2$, so the cycle length τ_1 is either 2 or 3, with mean $\mathbf{E}\tau_1 = 5/2$. The stochastic recursion is given by

$$X_{t+1} = \min(3, \max(0, X_t + \xi_t^{Z_t}))$$

where $\mathbf{P}(\xi_t^1 = k) = 1/4$ for $k = 0, 1, 2, 3$ and $\mathbf{P}(\xi_t^2 = -1) = \mathbf{P}(\xi_t^2 = -2) = 1/2$. It may be easily checked that the SRS satisfies all the conditions of the previous theorem.

Introduce the embedded Markov chain $Y_n = X_{T_n}$, as in the proof of the previous theorem. It is irreducible with transition probability matrix $P = \{p_{i,j}, 0 \leq i, j \leq 3\}$ given by

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 3/16 & 1/4 & 1/4 & 5/16 \\ 3/32 & 3/16 & 1/4 & 15/32 \end{pmatrix}.$$

For example, here

$$\begin{aligned} p_{3,1} &= \mathbf{P}(\tau_1 = 2, \xi_1^2 = -2, \xi_2^1 = 0) + \mathbf{P}(\tau_1 = 3, \xi_1^2 = \xi_2^2 = -1, \xi_3^1 = 0) \\ &+ \mathbf{P}(\tau_1 = 3, \xi_1^2 + \xi_2^2 = -3, \xi_3^1 = 1) + \mathbf{P}(\tau_1 = 3, \xi_1^2 = \xi_2^2 = -2, \xi_3^1 = 1) \\ &= \frac{1}{16} + \frac{1}{32} + \frac{1}{16} + \frac{1}{32} = \frac{3}{16}. \end{aligned}$$

Then the distribution of Y_n converges to $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ which may be found by solving $\pi P = \pi$ with $\sum \pi_i = 1$. So we get $\pi = (29/160, 183/800, 1/4, 17/50)$. Furthermore, the limiting distribution for X_t is given by

$$\begin{aligned} \mu_k &= \frac{1}{\mathbf{E}\tau_1}(\mathbf{P}(Y^{(0)} = k) + \mathbf{P}(\max(0, Y^{(0)} + \xi_0^2) = k) \\ &+ \mathbf{P}(\max(0, Y^{(0)} + \xi_0^2 + \xi_1^2) = k, \tau_1 = 3)), \end{aligned}$$

for $k = 0, 1, 2, 3$, where $Y^{(0)} \sim \pi$. In particular, $\mu_3 = 2\pi_3/5$, $\mu_2 = \frac{2}{5}(\pi_2 + \pi_3/2)$, and $\mu_1 = \frac{2}{5}(\pi_1 + (\pi_2 + \pi_3)/2 + \pi_3/8) = \frac{2}{5}(\pi_1 + \pi_2/2 + 5\pi_3/8)$.

2.3. *The case where the governing sequence is Markov*

In the particular case where $\{Z_t\}$ is a Markov chain on a countable state space, Theorem 2 leads to the following corollary, which is important for the examples considered in the next section.

Corollary 1 *Assume again that the recursive sequence $\{X_t\}$ is defined by (5), and that the function f is monotone increasing in the first argument. Assume in addition that $\{Z_t\}$ is an aperiodic Markov chain on a countable state space with a positive recurrent atom at point z_0. Assume also that there exists a number $0 \leq c \leq 1$, positive integers N_1 and N_2 and sequences $z_{1,1}, \ldots, z_{N_1,1}$ and $z_{1,2}, \ldots, z_{N_2,2}$ such that $z_{N_1,1} = z_{N_2,2} = z_0$ and, for $i = 1, 2$, the following hold:*

$$p_i := \mathbf{P}(Z_j = z_{j,i}, \textit{ for } \ j = 1, \ldots, N_i \mid Z_0 = z_0) > 0$$

and that

$$\delta_1 := \mathbf{P}(\widetilde{X}^{(1)}_{N_1} \le c \mid Z_0 = z_0,\ Z_j = z_{j,1},\ j = 1, \ldots, N_1) > 0$$

and

$$\delta_2 := \mathbf{P}(\widetilde{X}^{(0)}_{N_2} \ge c \mid Z_0 = z_0,\ Z_j = z_{j,2},\ j = 1, \ldots, N_2) > 0.$$

Then the distribution of X_t converges in the uniform metric to a unique stationary distribution.

Proof: We have to show that Corollary 1 follows from Theorem 2. For that, we have to define a typical (say, first) regenerative cycle and show that all the conditions of Theorem 2 hold.

Assume that $Z_0 = z_0$, so $T_0 = 0$. Let $T_1 = \tau_1 = \min\{t > 0 : Z_t = z_0\}$, then the aperiodicity means that $G.C.D.\{t : \mathbf{P}(T_1 = t) > 0\} = 1$. Let $T_n = \sum_1^n \tau_j$ where τ_j are i.i.d. copies of τ_1.

Let the conditions of the Corollary hold, and let k_i be the number of occurrences of z_0 in the sequence $z_{j,i}$, for $i = 1, 2$. Let L be the *least common multiple* of k_1 and k_2,

$$L = \min\{l \ : \ l/k_1 \ \text{ and } \ l/k_2 \ \text{ are integers}\}.$$

Let α be a random variable that takes values 0 and 1 with equal probabilities and does not depend on any of the processes defined in the model. Then define a regenerative cycle as follows: $\widehat{T}_0 = 0$ and

$$\widehat{T}_1 = T_1\alpha + T_L(1 - \alpha).$$

That is, we suppose that our regenerative cycle is either a single cycle or a sum of L cycles, with equal probabilities. Then all the conditions of Theorem 2 hold (with $\widehat{T}_i$ in place of T_i). Indeed, condition (3) follows since it holds for τ_1, and since $\widehat{T}_1$ is not bigger than T_L, the sum of L copies of τ_1. Condition (4) follows because the set of all t such that $\mathbf{P}(\widehat{T}_1 = t) > 0$ includes the set of all t such that $\mathbf{P}(\tau_1 = t) > 0$ and, therefore,

$$G.C.D.\{t \ : \ \mathbf{P}(\widehat{T}_1 = t) > 0\} \le G.C.D.\{t \ : \ \mathbf{P}(\tau_1 = t) > 0\},$$

so, given aperiodicity, both greatest common divisors are equal to 1. Finally, ε_1 in (8) is not smaller than $\frac{1}{2}p_1\delta_1 > 0$ and, similarly, ε_2 in (9) is not smaller than $\frac{1}{2}p_2\delta_2 > 0$. *Q.E.D.*

Remark 6 For simplicity, we have assumed that the Markov chain in Corollary 1 is defined on a countable state space with a positive recurrent atom. However, Corollary 1 can be extended to the case of a driving Markov chain on a general state space provided a "Harris-type" condition is satisfied. Here we outline the conditions required. Consider again a recursive sequence $\{X_t\}$ with the function f monotone increasing in the first argument. Assume that there exists a measurable set A in the state space $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ that is *positive recurrent*:

$$T_1(z_0) = \min\{t > 0 \ : \ Z_t^{(z_0)} \in A\} < \infty \quad \text{a.s., for any} \quad z_0 \in \mathcal{Z}$$

and

$$\sup_{z_0 \in A} \mathbf{E} T_1(z_0) < \infty.$$

Here $Z_t^{(z)}$ is a Markov chain with initial value $Z_0^{(z_0)} = z_0$. Furthermore, assume that there exist positive integers N_1 and N_2, positive number $p \le 1$ and a probability measure φ on A such that, for $i = 1, 2$ and all $z_0 \in A$,

$$\mathbf{P}(Z_{N_i}^{(z_0)} \in \cdot) \ge p\varphi(\cdot)$$

and that there exists a number $0 \le c \le 1$ and positive numbers δ_1 and δ_2 such that

$$\mathbf{P}(\widetilde{X}_{N_1}^{(1)} \le c \mid Z_0 = z_0, Z_{N_1} = z_1) \ge \delta_1$$

and

$$\mathbf{P}(\widetilde{X}_{N_2}^{(0)} \ge c \mid Z_0 = z_0, Z_{N_2} = z_2) \ge \delta_2,$$

for φ-almost surely all $z_0, z_1, z_2 \in A$. With all these conditions and aperiodicity of the Markov chain, it can be shown that the distribution of X_t converges in the uniform metric to the unique stationary distribution.

3. Applications

In this section we present three workhorse models. In each case, we assume that the driving process is a Markov chain and apply our Corollary 1. We are thus able to extend known stability results in these models.

3.1. *Bewley-Imrohoroglu-Huggett-Aiyagari Precautionary Savings Model*

The basic income fluctuation model in which many risk-averse agents self-insure against idiosyncratic income shocks through borrowing and saving using a risk-free asset, is designated by Heathcote et al. (2009) "the standard incomplete markets model" and is the workhorse model in quantitative macroeconomics. At its heart is the stochastic savings models of Huggett (1993) with an exogenous borrowing constraint, or close variants of this model.[8] As Heathcote et al. (2009) observe, there are "few general results that apply to this class of

[8]Bewley (1987) and Aiyagari (1994) vary the context but the individual savings problem is similar. They each derive existence and convergence results under slightly different assumptions. Bewley (1987) assumes that the endowment shocks are stationary Markov, Huggett (1993) assumes positive serial correlation and two states, and Aiyagari (1994) assumes that endowment shocks are i.i.d. İmrohoroğlu (1992) uses numerical methods with a two state persistent income process as in Huggett (1993).

problems." Existing analytical work in the standard model requires, to the best of our knowledge, either that income fluctuations are i.i.d., or that an individual's income process is persistent: a higher income today implies that income tomorrow is higher in the stochastic dominance sense. That is, that the income process is monotone. Huggett (1993) has a two-state process for income and uses the Hopenhayn and Prescott (1992) approach to prove convergence of the asset distribution to a unique, invariant distribution.[9] In the case of two income states, the assumption of persistence in the income process is probably innocuous. However, it may be restrictive in other cases. Obvious examples of non-monotone processes would include termination pay where a worker receives a large one-off redundancy payment followed by a long spell of unemployment, or health shocks where an insurance payout is received but future employment prospects are diminished.[10] In what follows we consider Huggett's model with a finite number of states (we maintain all his other assumptions). Applying our methodology, we show convergence to a unique, invariant distribution whilst dispensing with the assumption that the income process is monotone.[11]

Agents maximize expected discounted utility

$$\mathbf{E}\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right],$$

where $c_t \in \mathbb{R}_+$ is consumption at time t, $t = 0, 1, \dots,$ $u(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma > 1$, subject to a budget constraint at each date

$$c + R^{-1}a^+ \leq a + e,$$

a borrowing constraint $a^+ \geq \underline{a}$, where e is the current endowment, a is current assets, a^+ is assets next period, c is consumption, $\beta \in (0,1)$ is the discount factor and $R^{-1} > \beta$ is the price of next-period assets. The current endowment is drawn from a finite set $E := \{e^1, \dots, e^n\}$, $n \geq 2$, $e^n > e^1$, $e^{i+1} \geq e^i > 0$. all $i = 1, \dots, n-1$ (here superscripts are used to denote a particular realization of the endowment). The individual's endowment e_t is governed by a Markov chain

[9] Miao (2002) extends Huggett's model from two states to many states.

[10] As another example, consider the case where there are a group of entrepreneurs who have very high income. It may be possible that these entrepreneurs have a higher chance to fall to very low income levels than those on medium income levels. This is the situation described by Kaymak and Poschke (2015) who use information from observed distributions of income and wealth to construct a transition matrix for income. The transition matrix they use does not satisfy monotonicity.

[11] In the subsequent analysis, we follow Huggett and assume that the gross interest rate, R, is fixed. In a general equilibrium, there is a gross interest rate, R^* say, where assets are in zero net demand. The Bewley-Imrohoroglu-Huggett-Aiyagari literature is concerned with finding a stationary distribution of assets at the equilibrium R^*. However, uniqueness of the invariant distribution at any given R is important for continuity of the asset excess demand function. Equally, stability of the distribution is important both because it implies a straightforward approach for computing net asset demand at any candidate R (weak convergence is sufficient), and also as a heuristic justification for focusing on the steady state.

with stationary transition probabilities $p(e, e^+) := \mathbf{P}(e_{t+1} = e^+ \mid e_t = e) > 0$, for all $e, e^+ \in E$. The borrowing constraint satisfies $\underline{a} < 0$ and $\underline{a} + e^1 - \underline{a}R^{-1} > 0$. The initial values $e_0 \in E$ and $a_0 \geq \underline{a}$ are given.

The individual's decision problem can be represented by the functional equation:

$$v(a, e) = \max_{(c, a^+) \in \Gamma(a, e)} u(c) + \beta \mathbf{E}_{e^+} \left[v(a^+, e^+) \mid e \right] \tag{10}$$

where $\mathbf{E}_{e^+}$ is expectation over e^+, $v(a, e)$ are the value functions (one for each realization e), and

$$\Gamma(a, e) = \left\{ (c, a^+) \mid c + R^{-1} a^+ \leq a + e, a^+ \geq \underline{a}, c \geq 0 \right\}$$

is the constraint set. The resulting policy functions are denoted $c = c(a, e)$ and $a^+ = f(a, e)$. Huggett (Theorem 1) proves that there is a unique, bounded and continuous solution to (10) and each $v(a, e)$ is increasing, strictly concave and continuously differentiable in a, while f is continuous and nondecreasing in a, and (strictly) increasing whenever $f(a, e) > \underline{a}$. These results extend to our context with more than two endowment states; see Miao (2002).

Huggett assumes monotonicity or persistence of the endowment process: with $n = 2$ this means $p(e^1, e^1) \geq p(e^2, e^1)$. He shows that for a given R, there exists a unique stationary probability measure for $x = (a, e)$ and that there is weak convergence to this distribution for any initial distribution on x (Huggett [Theorem 2]).

We can extend this result to our more general context using the following (the proof can be found in the Appendix).[12]

Lemma 1 *(i)For $a > \underline{a}$, there is at least one $e \in E$ such that $f(a, e) < a$; (ii) there exists $\widehat{a} \geq \underline{a}$ such that for all $a > \widehat{a}$, all $e \in E$, $f(a, e) < a$.*

Define $\bar{a} := \min_a \{a \geq \underline{a} : f(a, e) \leq a \text{ all } e \in E\} < \infty$. (This exists by part (ii) of the lemma, given the continuity of $f(\cdot, e)$.) If $\bar{a} > \underline{a}$, then define $\underline{f}(a) := \min_e \{f(a, e)\}$, where $\underline{f}(a) < a$ on $(\underline{a}, \bar{a}]$ by part (i) of the lemma, while $\overline{f}(a) := \max_e \{f(a, e)\} > a$ on $[\underline{a}, \bar{a})$ by definition of $\bar{a}$. Starting from $(\bar{a}, e_0)$, repeatedly applying $\underline{f}$ yields the strictly decreasing sequence $(\underline{f}(f(\bar{a}, e_0)), \underline{f}^{(2)}(f(\bar{a}, e_0)), \ldots)$ where $\underline{f}^{(n)}$ denotes the n-fold composition of $\underline{f}$. Suppose that $\lim_{T\to\infty} \underline{f}^{(T)}(f(\bar{a}, e_0)) = \widetilde{a} > \underline{a}$. Then by continuity of $\underline{f}$, $\underline{f}(\widetilde{a}) = \widetilde{a}$, which contradicts part (i) of the lemma. Thus, $\lim_{T\to\infty} \underline{f}^{(T)}(f(\bar{a}, e_0)) = \underline{a}$, and fixing some $a^+ \in (\underline{a}, \bar{a})$, there exists a finite sequence $(e_t)_{t=1}^T$ with $e_t \in \arg\min_{e \in E} \{f(\underline{f}^{(t-1)}(f(\bar{a}, e_0)), e)\}$ such that the occurrence of $(e_t)_{t=1}^T$ implies $a_t < a^+$. Moreover, the sequence $(e_t)_{t=1}^T$ has positive probability since all the transition probabilities are positive. By a symmetric

[12]Similar properties are established in Huggett (1993), and in Miao (2002) for the many state case, but using the persistence assumption.

argument, using $\bar{f}(a)$ and starting from $(\underline{a}, e_0)$, there exists a positive probability, finite sequence of endowment shocks $(\widetilde{e}_t)_{t=1}^{\widetilde{T}}$ whose occurrence implies $a_t > a^+$. Let $\{Z_t\}$ in Corollary 1 be identified with the process $\{e_t\}$, assuming that each $\xi_t^{Z_t}$ has a degenerate distribution, so that $f(X_t, \xi_t^{Z_t}) = f(a_t, e_t)$. Setting $z_0 = e_0$, $c = a^+$, $z_{j,2} = e^j$, $N_2 = T - 1$, $z_{j,1} = e^j$, $N_1 = \widetilde{T} - 1$, and given $p(e^i, e^j) > 0$ for each i and j, the conditions of the corollary are all satisfied.

This implies that there exists a unique distribution π such that the distributions of a_t converge to π in the uniform metric for any initial value $a_0 \in [\underline{a}, \bar{a}]$. (If $\bar{a} = \underline{a}$ then this would be a degenerate limit.) Moreover, any subset of $(\bar{a}, \infty)$ is transient.

3.2. *One-Sector Stochastic Optimal Growth Model*

The Brock-Mirman (Brock and Mirman, 1972) one-sector stochastic optimal growth model has been extended to the case of correlated production shocks by Donaldson and Mehra (1983) and Hopenhayn and Prescott (1992, pp. 1402–03). With correlated productivity shocks, it is possible to prove uniqueness and convergence results using the methods of Hopenhayn and Prescott (1992) or Stokey et al. (1989, Chapter 12) provided the policy function for the investment is itself monotonic in the productivity shock. Although the assumption of correlated shocks is very reasonable in this context, establishing that the policy function is monotone in the productivity shock is, as pointed out by Hopenhayn and Prescott (1992, pp. 1403), difficult without imposing very restrictive assumptions. The reason is simple. A good productivity shock today increases current output, which may allow increased investment. However, because shocks are positively correlated, output will also be higher on average tomorrow and hence consumption can be too. Therefore, it may be desirable to increase current consumption by more than the increase in current output, cutting back on current investment.[13] Since our results do not require monotonicity of the policy function in the driving process, we can establish convergence to a unique invariant distribution without requiring any extra restrictive conditions on preferences and productivity beyond those normally assumed in the stochastic growth model. In addition, of course, we do not require the productivity shocks to be positively correlated.

[13] The sufficient condition given in Hopenhayn and Prescott (1992) for monotonicity of the policy function in the productivity shock is

$$\frac{\phi_{kz}}{\phi_k \cdot \phi_z} \geq -\frac{u''}{u'},$$

where ϕ is the production function, depending on capital k and productivity shock z, and u is the utility function. Since the arguments of the utility function and production function depend on the policy function themselves, this condition is difficult to check a priori, except in special cases. One such special case is where the capital and productivity shock are perfect complements in production, in which case the left-hand-side of the above inequality becomes infinitely large.

We consider a version of the Brock-Mirman one sector stochastic optimal growth model with full depreciation of capital. Paths for consumption and capital are chosen to

$$\max \mathbf{E} \sum\nolimits_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\phi(k_t, z_t) \geq c_t + k_{t+1}, \quad c_t \geq 0,$$

with $k_0 \geq 0$ and $z_0 \in \widehat{\mathcal{Z}}$ given. The utility function $u: \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing, and strictly concave with $\lim_{c\to 0} u'(c) = \infty$. The production function $\phi: \mathbb{R}_+ \times \widehat{\mathcal{Z}} \to \mathbb{R}_+$, with ϕ continuously differentiable, strictly increasing and strictly concave in k with $\lim_{k\to 0} \phi_k(k, z) = \infty$ (where ϕ_k denote $\partial\phi(k, z)/\partial k$), where there exists a $k^{\max} > 0$ such that $\phi(k, z) < k$ for all $k > k^{\max}$ and all $z \in \widehat{\mathcal{Z}}$. The discount factor $0 < \beta < 1$. The productivity shock is drawn from a finite set $\hat{\mathcal{Z}} := \{z^1, \ldots, z^n\}$, $n \geq 2$, with z_t governed by a Markov chain with stationary transition probabilities $p(z, z^+) := \mathbf{P}(z_{t+1} = z^+ \mid z_t = z) > 0$, for all $z, z^+ \in \widehat{\mathcal{Z}}$. We assume:

Assumption A. (i) $\phi(0, \tilde{z}) > 0$ for some $\tilde{z} \in \widehat{\mathcal{Z}}$; (ii) there exist z', $z'' \in \widehat{\mathcal{Z}}$, such that $\phi(k, z') > \phi(k, z'')$, $\forall k > 0$, and $p(z', z) = p(z'', z)$ for all $z \in \widehat{\mathcal{Z}}$.

Part (i) of the assumption is a simple way to avoid $k = 0$ being a trivial stable point.[14] Part (ii) of the assumption means there are at least two states for which output is strictly higher in one but which do not affect the distribution over future states. This is a simple way to guarantee that choice of the next period capital stock differs for at least two of the possible realizations of z.[15]

The problem can be set up recursively. Letting k^+ denote next period's capital stock and z^+ next period's shock, the value function satisfies

$$v(k, z) = \max_{0 \leq k^+ \leq \phi(k,z)} u(\phi(k, z) - k^+) + \beta \mathbf{E}_{z^+} \left[v(k^+, z^+) \mid z \right] \tag{11}$$

where $\mathbf{E}_{z^+}$ is expectation over z^+. Let $k_{t+1} = f(k_t, z_t)$ be the policy function, and $c(k, z) := \phi(k, z) - f(k, z)$. The following is standard (see e.g. Stokey et al., 1989, ch. 10): $f(k, z)$ is continuous and increasing in k; moreover $v(k, z)$ is increasing, strictly concave and differentiable in k for $k > 0$.

For $k > 0$, the solution to the maximization problem in (11) is interior.[16] Thus,

[14] See Stokey et al. (1989, p.392) for a similar assumption. There are a variety of alternative approaches that imply convergence on a set $[\varepsilon, k^{\max}]$ where $\varepsilon > 0$, and these could be adapted to our context.

[15] Alternatively, we could allow for more general distributions of random variables $\{\xi_t^z\}$ as in Section 2, and such that output varies with ξ_t^z.

[16] We have $k^+ > 0$ because the marginal return to saving, $\beta \mathbf{E}_{z^+}[u'(c^+)\phi_k(k^+, z^+) \mid z] \to \infty$ as $k^+ \to 0$ which therefore exceeds $u'(c_t)$ for all k^+ near zero. Similarly, the condition $\lim_{c\to 0} u'(c) = \infty$ ensures $k^+ < \phi(k, z)$.

the first-order and envelope conditions are given by:

$$u'(c(k,z)) = \beta \mathrm{E}_{z^+}\left[v_k(f(k,z),z^+)) \mid z\right], \tag{12}$$

$$v_k(k,z) = u'(c(k,z))\phi_k(k,z). \tag{13}$$

Combining (12) and (13), we have:

$$v_k(k,z) = \beta\phi_k(k,z)\mathbf{E}_{z^+}\left[v_k(f(k,z),z^+)) \mid z\right]. \tag{14}$$

To examine convergence, define $k'' := \min\{k \mid \max_z f(k,z) \le k\}$. This must exist by continuity of f, $f(0,\tilde{z}) > 0$ by Assumption A, and $f(k,z) \le \phi(k,z) < k$ for all $k > k^{\max}$, all z. Similarly, define $k' := \max\{k \le k'' \mid \min_z f(k,z) = k\}$. Note that $k'' > k' \ge 0$ by continuity of f and Assumption A which implies[17] $f(k,z') > f(k,z'')$ for $k > 0$, and hence, that $\max_z f(k'',z) = k'' > \min_z f(k'',z)$. To establish convergence, we use the following lemma (the proof can be found in the Appendix).

Lemma 2 $\forall k > k'$, $\min_z f(k,z) < k$.

Then take any $k \in (k',k'')$. By the lemma, and by definition of k'', $\max_z f(k,z) > k$ for all $k < k''$. Therefore, we can repeat, *mutatis mutandis*, the argument of the previous subsection to establish that the mixing condition is satisfied on $[0,k^{\max}]$ and our convergence result applies: there exists a unique stationary distribution π such that the distributions of k_t converge to π in the uniform metric for any initial value $k_0 \in [0,k^{\max}]$.

3.3. *Limited Commitment Risk-Sharing Model*

In this section we consider the inter-temporal risk-sharing model with limited commitment. Kocherlakota (1996) (see also, for example, Alvarez and Jermann, 2000, 2001; Ligon et al., 2002; Thomas and Worrall, 1988) provides a convergence result for the long-run distribution of risk-sharing transfers when shocks to income are finite and i.i.d. His model has two, infinitely-lived, risk averse agents with per-period, strictly concave and differentiable utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ defined over consumption, and a common discount factor β. Agent 1 has a random endowment $y_t > 0$ at date $t = 0,1,\ldots,$ and agent 2 has a random endowment $Y - y_t > 0$ where $Y > 0$ is a constant aggregate income. The endowment shock is drawn from a finite set $\mathcal{Y} := \{y^1,\ldots,y^n\}$, $n \ge 2$, with y_t governed by a Markov chain with stationary transition probabilities $p(y,y^+) := \mathbf{P}(y_{t+1} = y^+ \mid y_t = y) > 0$, for all $y, y^+ \in \mathcal{Y}$. There is no credit

[17] If $f(k,z') \le f(k,z'')$, then $c(k,z') > c(k,z'')$ by $\phi(k,z') > \phi(k,z'')$. This leads to a contradiction of (12). The LHS of (12) is strictly lower at z' than at z'' by the concavity of the utility function. Conversely, by the concavity of v, $v_k(g(k,z'),z^+) \ge v_k(g(k,z''),z^+)$, meaning the the RHS of (12) is no lower at at z' than at z'', because the future distribution of the shock is the same for both z' and z'' by Assumption A.

market but agents can transfer income between themselves at any date. Although Kocherlakota (1996) assumes the endowment shocks are i.i.d., we will show that this convergence result is easily extended to the case where y_t is a Markov chain. Letting $h^t = (y_0, y_1, \ldots, y_t)$ denote the history of income realizations, agents choose a sequence of history-dependent transfers $X_t(h^t)$ from agent 1 to agent 2 subject to $-Y + y_t \le X_t(h^t) \le y_t$ for each h^t and the self-enforcing constraints that neither agent prefers autarky from that point on after any history over the agreed transfer plan. In particular, the self-enforcing constraints for the two agents are

$$\begin{aligned} u(y_t - X_t(h^t)) &+ \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(y_{t+s} - X_t(h^{t+s}))] \\ &\ge u(y_t) + \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(y_{t+s}))], \\ u(Y - y_t + X_t(h^t)) &+ \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(Y - y_{t+s} + X_t(h^{t+s}))] \\ &\ge u(Y - y_t) + \mathbf{E}_t[\sum_{s=1}^{\infty} \beta^s u(Y - y_{t+s}))], \end{aligned}$$

for each date t and h^t. An *efficient risk-sharing arrangement* will solve (for some feasible U^0):

$$\max_{\{X_t\}} \mathbf{E}_0[\sum_{s=0}^{\infty} \beta^s u(y_s - X_s(h^s))] \quad \text{s.t.} \quad \mathbf{E}_0[\sum_{s=0}^{\infty} \beta^s u(Y - y_s + X_s(h^s))] \ge U^0.$$

and subject to the self-enforcing constraints. It is well known (see, e.g., Ligon et al., 2002) that the solution at each date has the following property: For each realization y, there is a time-invariant interval $I_y = [\underline{c}_y, \overline{c}_y]$, $\underline{c}_y \le \overline{c}_y$, such that

$$c_{t+1}\left(h^{t+1}\right) := y_{t+1} - X_{t+1}(h^{t+1}) = \begin{cases} \overline{c}_{y_{t+1}} & \text{if } c_t(h^t) > \overline{c}_{y_{t+1}} \\ c_t(h^t) & \text{if } c_t(h^t) \in I_{y_{t+1}} \\ \underline{c}_{y_{t+1}} & \text{if } c_t(h^t) < \underline{c}_{y_{t+1}} \end{cases},$$

and there is a one-to-one correspondence between feasible U^0 and agent 1's initial consumption $c_0(h^0) \in [\underline{c}_{y_0}, \overline{c}_{y_0}]$. We can write this in the form (5) as $c_{t+1} = f(c_t, \xi_t)$ where $\xi_t := y_{t+1}$, and where

$$f(c, \xi) = \begin{cases} \overline{c}_\xi & \text{if } c > \overline{c}_\xi \\ c & \text{if } c \in I_\xi \\ \underline{c}_\xi & \text{if } c < \underline{c}_\xi \end{cases}.$$

The first-best is sustainable for some U^0 if and only if $\cap_\xi I_\xi \neq \emptyset$. Kocherlakota (1996) shows (his Proposition 4.2) that if shocks are i.i.d. and if the first-best is not sustainable then the distribution of transfers converges weakly to the same non-degenerate distribution for all U^0. We now show how to easily extend this result to the case where shocks follow a Markov chain. Define $c_{\min} := \min_\xi \overline{c}_\xi$, $c_{\max} := \max_\xi \underline{c}_\xi$. If the first-best is not sustainable, $\cap_\xi I_\xi = \emptyset$, then $c_{\min} < c_{\max}$. If $c_t \in [c_{\min}, c_{\max}]$, $c_{t+1} = f(c_t, \xi_t) \in [c_{\min}, c_{\max}]$ for all ξ_t. Define $c := (c_{\min} + c_{\max})/2$. Using the notation of Corollary 1 (where $[c_{\min}, c_{\max}]$ replaces $[0,1]$), let $N_1 = N_2 = 2$, $z_{1,1} \in \arg\max_\xi \underline{c}_\xi$, $z_{1,2} \in \arg\min_\xi \overline{c}_\xi$. For any z_0, all the assumptions of the corollary are satisfied. Thus, there exists a unique distribution π such that the distributions of c_t converge to π in the uniform metric for any initial value $c_0 \in [c_{\min}, c_{\max}]$. Clearly, $c_t \in [\underline{c}_{\xi_0}, \overline{c}_{\xi_0}] \cup [c_{\min}, c_{\max}]$ all t, and $[\underline{c}_{\xi_0}, \overline{c}_{\xi_0}] \backslash [c_{\min}, c_{\max}]$ is transient.

If the first-best is sustainable, then the mixing condition is not satisfied. In that case it can be seen immediately that there is monotone convergence to a first-best allocation (the limit allocation is dependent on the initial condition).

4. CONCLUSION

In this paper we have established convergence results that can be used in a range of models whose dynamics can be represented by a stochastic recursion, and which satisfy two main conditions; first, for a given value of the exogenous driving process, the future value of the endogenous variable is monotone increasing in its current value; secondly, the driving process is regenerative. The latter includes as a special case irreducible finite Markov chains. These two conditions, along with a standard mixing condition, guarantee weak convergence to a unique stationary distribution.

This extends the existing results on convergence of monotone Markov processes that assume the driving process is i.i.d. or assume that the driving process is itself a monotone Markov process (Hopenhayn and Prescott, 1992). This extension is important because most economic models take the driving process for the underlying shocks to be exogenous and therefore it is useful to have results for a broader class of stochastic driving processes. Moreover, we do not require that the stochastic recursion is monotone in the second argument. This is particularly useful when the stochastic recursion is derived as a policy function of a dynamic programming problem because establishing monotonicity in the shock process might require extra restrictions on preferences or technology.

We have applied our approach to three workhorse models in macroeconomics extending our understanding of stability in these models. Our Theorem 2 and its corollary can also be readily used to establish convergence to a unique stationary distribution for any monotone stochastic recursion in a regenerative environment where the appropriate mixing condition is satisfied.

APPENDIX

Proof of Theorem 1.

PROOF: The metric space of probability distributions on $[0,1]$ with metric d is complete. Due to monotonicity, it is sufficient to show that

$$d(F_t^{(0)}, F_t^{(1)}) \to 0$$

exponentially fast. Then (7) will follow.

Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Denote by A and B the events

$$A = \{X_N^{(1)} \leq c\} \quad \text{and} \quad B = \{X_N^{(0)} \geq c\}.$$

Note that both events are defined by $(\xi_0, \dots, \xi_{N-1})$, i.e. belong to the sigma-algebra generated by these random variables.

The proof is by induction. For any $r \geq c$ and for any two probability measures μ and ν on $[0,1]$ with $\mu(x) \equiv \mu[0,x] \geq \nu(x) \equiv \nu[0,x]$, for all x, we may couple initial values of 4 trajectories of the Markov chain $\{X_t^{(1)}\}, \{X_t^{(\nu)}\}, \{X_t^{(\mu)}\}$, $\{X_t^{(0}\}$ in such a way that

$$1 = X_0^{(1)} \geq X_0^{(\nu)} \geq X_0^{(\mu)} \geq X_0^{(0} = 0 \quad \text{a.s.}$$

Then

$$X_t^{(1)} \geq X_t^{(\nu)} \geq X_t^{(\mu)} \geq X_t^{(0)} \quad \text{a.s. for any} \quad t,$$

and we have

$$\begin{aligned}
0 \leq F_N^{(\mu)}(r) - F_N^{(\nu)}(r) &= \mathbf{P}(X_N^{(\mu)} \leq r, A) + \mathbf{P}(X_N^{(\mu)} \leq r, \overline{A}) \\
&- \mathbf{P}(X_N^{(\nu)} \leq r, A) - \mathbf{P}(X_N^{(\nu)} \leq r, \overline{A}) \\
&= \mathbf{P}(A) + \mathbf{P}(X_N^{(\mu)} \leq r, \overline{A}) \\
&- \mathbf{P}(A) - \mathbf{P}(X_N^{(\nu)} \leq r, \overline{A}) \\
&= \int_{\overline{A}} (\mu(S^{(N)}(\overline{v}, r)) - \nu(S^{(N)}(\overline{v}, r)))\mathbf{P}((\xi_0, \dots \xi_{N-1}) \in d\, \overline{v}) \\
&\leq \sup_x(\mu(x) - \nu(x)) \cdot \mathbf{P}(\overline{A}) \\
&\leq (1-\varepsilon) \sup_x(\mu(x) - \nu(x)).
\end{aligned}$$

Similarly, for $r < c$, we may use event B to conclude again that

$$0 \leq F_N^{(\mu)}(r) - F_N^{(\nu)}(r) \leq (1-\varepsilon) \sup_x(\mu(x) - \nu(x)).$$

Therefore,

$$\sup_r(F_N^{(\mu)}(r) - F_N^{(\nu)}(r)) \leq (1-\varepsilon) \sup_x(\mu(x) - \nu(x)).$$

Now, by induction, we obtain

$$0 \leq F_{kN}^{(0)}(r) - F_{kN}^{(1)}(r) \leq (1-\varepsilon)^k$$

for all r.

Indeed, for $k = 1$ the inequality follows from the above. Assume that it holds for $k \leq K - 1$. Then

$$\begin{aligned} 0 &\leq F_{KN}^{(0)}(r) - F_{KN}^{(1)}(r) = \mathbf{P}\left(X_{KN}^{(0)} \leq r\right) - \mathbf{P}\left(X_{KN}^{(1)} \leq r\right) \\ &= \mathbf{P}\left(X_N^{(X_{(K-1)N}^{(0)})} \leq r\right) - \mathbf{P}\left(X_N^{(X_{(K-1)N}^{(1)})} \leq r\right) \\ &\leq (1-\varepsilon)\sup_r \left(F_{(K-1)N}^{(0)}(r) - F_{(K-1)N}^{(1)}(r)\right) \leq (1-\varepsilon)^K, \end{aligned}$$

which finishes the proof of the induction argument, and the result now follows. *Q.E.D.*

Proof of Lemma 1.

PROOF: (i) For $a > \underline{a}$, we cannot have $f(a,e) \geq a$ all $e \in E$. That is, assets will be reduced in at least one state. Suppose otherwise. Then for some $a > \underline{a}$ and for all e, where we write $v_a(a,e)$ for $\partial v(a,e)/\partial a$:

$$\begin{aligned} v_a(a,e) &= u'(c(a,e)) \\ &= \beta R \mathbf{E}_{e^+}\left[v_a(f(a,e),e^+) \mid e\right] \\ &\leq \beta R \mathbf{E}_{e^+}\left[v_a(a,e^+) \mid e\right] \\ &< \mathbf{E}_{e^+}\left[v_a(a,e^+) \mid e\right], \end{aligned} \tag{A.1}$$

where the first-line is the envelope condition, the second line follows from the first-order conditions to (10) (equality because $f(a,e) > \underline{a}$), the third line from the assumption that $f(a,e) \geq a$, which from the concavity of $v(\cdot, e)$ in a implies $v_a(f(a,e),e^+) \leq v_a(a,e^+)$ for all e^+, and the final line since $\beta R < 1$ and $v_a(a,e^+) > 0$. This is a contradiction because (A.1) cannot hold for all e (consider $e \in \arg\max_{e^+} v_a(a,e^+)$).

(ii) Suppose that, at some (a,e) with $a > \underline{a}$, $f(a,e) \geq a$. If the same shock recurs at the next date we have $c(f(a,e),e) \geq c(a,e)$ by c increasing in its first argument, which in turn follows from $v_a(a,e) = u'(c(a,e))$ and v (respectively u) strictly concave in a (c). So consumption is at least as high, and we shall show that assets also do not decline at this date; we show that in order to satisfy the Euler equation there must be some other state with consumption sufficiently low that assets will increase after this state too, and moreover marginal utility will also have risen; by repeating this construction a contradiction will arise. Since $f(a,e) > \underline{a}$, the Euler equation holds with equality (repeating lines 1 and 2 of (A.1)):

$$u'(c(a,e)) = \beta R \mathbf{E}_{e^+}\left[u'(c(f(a,e),e^+)) \mid e\right]. \tag{A.2}$$

From $c(f(a,e),e) \geq c(a,e)$, $\beta R < 1$ and u being strictly concave, (A.2) implies that $u'(c(a,e)) \leq \beta R u'(c(f(a,e),e^+))$ for some $e^+ \neq e$. Thus

$$c(f(a,e),e^+)^{-\gamma} \geq (\beta R)^{-1} c(a,e)^{-\gamma}$$

so

$$c(f(a,e),e^+) \leq (\beta R)^{1/\gamma} c(a,e). \tag{A.3}$$

For

$$c(a,e) > (e^n - e^1)/(1 - (\beta R)^{1/\gamma}), \tag{A.4}$$

we have from (A.3):

$$\begin{aligned} c(a,e) - c(f(a,e),e^+) &\geq (1 - (\beta R)^{1/\gamma}) c(a,e) \\ &> (e_n - e_1). \end{aligned} \tag{A.5}$$

Moreover, $f(a,e) \geq a$ implies $f(f(a,e),e) \geq f(a,e)$, by $f(\cdot,e)$ nondecreasing in a, that is

$$f(f(a,e),e) = R(f(a,e) + e - c(f(a,e),e)) \geq f(a,e). \tag{A.6}$$

Then if (A.4) holds,

$$\begin{aligned} f(f(a,e),e^+) &= R(f(a,e) + e^+ - c(f(a,e),e^+)) \\ &> R(f(a,e) + e^+ + (e^n - e^1) - c(a,e)) \\ &\geq R(f(a,e) + e^+ + (e^n - e^1) - c(f(a,e),e)) \\ &\geq f(a,e), \end{aligned} \tag{A.7}$$

where the first line follows from the budget constraint, the second from (A.5), the third from $c(f(a,e),e) \geq c(a,e)$, and the last from (A.6) given that $e - e^+ \leq (e^n - e^1)$.

From v bounded in a and the envelope condition $v_a(a,e) = u'(c(a,e))$, we have $\lim_{a\to\infty} c(a,e) \to \infty$, so there exists some $\widehat{a}$ such that (A.4) holds for $a > \widehat{a}$, all $e \in E$. We conclude that if $a_0 > \widehat{a}$ and $f(a_0,e_0) \geq a_0$ for some realization $e_0 \in E$ at date $t = 0$, we have $f(f(a_0,e_0),e_1) > f(a_0,e_0)$ for some $e_1 \neq e_0$ from (A.7). Since $f(a_0,e_0) > \widehat{a}$, we can repeat the same argument in state e_1 with assets $a_1 = f(a_0,e_0)$, and so on. This implies a sequence $(a_0,e_0),(a_1,e_1),(a_2,e_2),\ldots$, with $a_{m+1} > a_m$ for $m \geq 1$, and with $u'(c(a_{m+1},e_{m+1})) \geq (\beta R)^{-1} u'(c(a_m,e_m))$ for $m \geq 0$. Clearly $e_m = e_p$ for some $1 \leq m < p \leq n+1$. Then

$$u'(c(a_p,e_m)) \geq (\beta R)^{-1} u'(c(a_m,e_m)),$$

so that $c(a_p,e_m) < c(a_m,e_m)$, which contradicts $c(a,e)$ increasing in its first argument. *Q.E.D.*

Proof of Lemma 2.

PROOF: Suppose not; let $\hat{k} > k'$ be such that $\min_z f(\hat{k}, z) = \hat{k}$. By definition of k', and $k'' > \min_z f(k'', z)$ (see above), $\hat{k} > k''$.

Consider any $(k, z) \in [k', k''] \times \widehat{\mathcal{Z}}$. Then $f(k, z) \in [k', k'']$ since $f(k, z) \geq \min_{z'} f(k, z') \geq \min_{z'} f(k', z') = k'$ where the second inequality follows from f increasing in k, and the equality from the definition of k'; likewise, $f(k, z) \leq \max_{z'} f(k, z') \leq \max_{z'} f(k'', z') = k''$ where the second inequality follows from f increasing in k, and the equality from the definition of k''.

Similarly, for $k \geq \hat{k}$, $f(k, z) \geq \hat{k}$, $\forall z \in \widehat{\mathcal{Z}}$, since $f(k, z) \geq \min_{z'} f(k, z') \geq \min_{z'} f(\hat{k}, z') = \hat{k}$.

We shall demonstrate a contradiction. Take any $(\bar{k}, z_0) \in (k', k'') \times \widehat{\mathcal{Z}}$, and define recursively

$$\begin{aligned} &\bar{k}_0 = \bar{k}; \\ &\bar{k}_\tau = f(\bar{k}_{\tau-1}, z_{\tau-1}) \qquad \tau = 1, \ldots, N. \end{aligned}$$

Iterating (14) $N > 0$ times:

$$v_k(\bar{k}, z) = \mathbf{E}\left[\beta^N \Pi_{\tau=0}^{N-1} \phi_k(\bar{k}_\tau, z_\tau) v_k(\bar{k}_N, z_N) \mid z_0\right]. \tag{A.8}$$

Likewise, for any $\tilde{k} \geq \hat{k}$, defining $\tilde{k}_\tau$ (analogously to $\bar{k}_\tau$) starting from $(\tilde{k}, z_0)$,

$$v_k(\tilde{k}, z) = \mathbf{E}\left[\beta^N \Pi_{\tau=0}^{N-1} \phi_k(\tilde{k}_\tau, z_\tau) v_k(\tilde{k}_N, z_N) \mid z_0\right]. \tag{A.9}$$

By $\bar{k}_\tau \in [k', k'']$, $\tilde{k}_\tau \geq \hat{k}$, $k'' < \hat{k}$, and the the strict concavity of ϕ and v in k :

$$\phi_k(\bar{k}_\tau, z_\tau) \geq \gamma \phi_k(\tilde{k}_\tau, z_\tau) \qquad a.s., \tag{A.10}$$

for some $\gamma > 1$, and

$$v_k(\bar{k}_N, z_N) > v_k(\tilde{k}_N, z_N) \qquad a.s. \tag{A.11}$$

Thus, from (A.8), (A.9), (A.10) and (A.11):

$$v_k(\bar{k}, z) > \gamma^N v_k(\tilde{k}; z).$$

Since $v_k(\bar{k}; z) < \infty$ by $\bar{k} > 0$, $\gamma > 1$, $v_k(\tilde{k}; z) > 0$, letting $N \to \infty$ yields a contradiction. *Q.E.D.*

Acknowledgement: The research of S. Foss was partially supported by EPSRC grant EP/I017054/1. The research of V. Shneer was supported by EPSRC grant EP/L026767/1. The Research of J. Thomas and T. Worrall was supported by ESRC grant ES/L009633/1.

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